Generating function associated with the Hankel determinant formula for the solutions of the Painlevé IV equation

N. Joshi, K. Kajiwara
M. Mazzocco

MHF 2006-14

(Received March 21, 2006)
Generating Function Associated with the Hankel Determinant Formula for the Solutions of the Painlevé IV Equation

Nalini JOSHI
School of Mathematics and Statistics F07, The University of Sydney,
NSW 2006, Australia
nalini@maths.usyd.edu.au

Kenji KAJIWARA
Graduate School of Mathematics, Kyushu University,
6-10-1 Hakozaki, Higashi-ku, Fukuoka 812-8512, Japan
kaji@math.kyushu-u.ac.jp

Marta MAZZOCCO
School of Mathematics, The University of Manchester,
Sackville Street, Manchester M60 1QD, United Kingdom.
marta.mazzocco@manchester.ac.uk

Abstract

We consider a Hankel determinant formula for generic solutions of the Painlevé IV equation. We show that the generating functions for the entries of the Hankel determinants are related to the asymptotic solution at infinity of the isomonodromic problem. Summability of these generating functions is also discussed.

Keywords and Phrases. Painlevé equation, determinant formula, isomonodromic problem
2000 Mathematics Subject Classification Numbers. 34M55, 34M25, 34E05

1 Introduction

In this article, we consider the Painlevé IV equation (P_{IV}),

$$\frac{d^2 y}{dt^2} = \frac{1}{2y} \left( \frac{dy}{dt} \right)^2 + \frac{3}{2} y^3 - 2ty^2 + \left( \frac{t^2}{2} - \alpha_1 + \alpha_0 \right) y - \frac{\alpha_2^3}{2y},$$

(1.1)

where $\alpha_i$ ($i = 0, 1, 2$) are parameters satisfying $\alpha_0 + \alpha_1 + \alpha_2 = 1$. We also denote Equation (1.1) as $P_{IV}[\alpha_0, \alpha_1, \alpha_2]$ when it is necessary to specify the parameters explicitly. $P_{IV}$ (1.1) has been studied extensively from various points of view. In particular, it is well-known that it admits symmetries of affine Weyl group of type $A_2^{(1)}$ as a group of Bäcklund transformations[15, 17, 18] (see also [13, 1, 3]). Moreover, it is known that $P_{IV}$ admits two classes of classical solutions for special values of parameters: transcendental classical solutions which are expressed by the parabolic cylinder function[18] and rational solutions [10, 14]. Otherwise the solutions are higher transcendental functions[16, 19]. Among these
solutions, particular attention has been paid to a class of rational solutions which are located on the center of the Weyl chambers. Those solutions are expressed by logarithmic derivatives of certain characteristic polynomials with integer coefficients, which are generated by the Toda equation. Those polynomials are called the Okamoto polynomials\[15, 17, 18, 4\].

The determinant formulas are useful for understanding the nature of the Okamoto polynomials. In fact, it has been shown that the Okamoto polynomials are nothing but a specialization of the 3-core Schur functions by using the Jacobi-Trudi type determinant formula \[12, 15, 17\]. Also, there is another determinant formula which expresses the Okamoto polynomials in terms of the Hankel determinant. Therefore it is an intriguing problem to clarify the underlying meaning of the Hankel determinant formula. In order to do this, generating functions for the entries of Hankel determinants have been constructed in \[9\], and it is shown that they are expressed as logarithmic derivatives of solutions of the Airy equation.

A similar phenomenon has been observed in the study of rational solution of the Painlevé II equation (P\(_\text{II}\))\[6\]. Namely, the generating function associated with the rational solutions of P\(_\text{II}\) is also expressed as logarithmic derivatives of the Airy function.

Then what do these phenomena mean? In order to answer the question, the Hankel determinant formula for the *generic* solution of P\(_\text{II}\) was considered in \[8\]. It was shown that the generating functions of the entries of the Hankel determinant formula are related to the solutions of isomonodromic problem of P\(_\text{II}\) \[7\]. More precisely, the coefficients of asymptotic expansion of the ratio of solutions of the isomonodromic problem at infinity give the entries of Hankel determinant formula. The next natural problem then is to investigate whether such structure can be seen in other Painlevé equations or not.

The purpose of this article is to study the generating functions associated with the Hankel determinant formula for the generic solutions of P\(_\text{IV}\) (1.1). In Section 2, we give a brief review of the symmetries and \(\tau\) functions for P\(_\text{IV}\) through the theory of the symmetric form of P\(_\text{IV}\)[15]. We then construct the Hankel determinant formula by applying the formula for the Toda equation\[11\] in Section 3. In Section 4, we consider the generating functions for the entries of the Hankel determinant formula. By linearizing the Riccati equations satisfied by the generating functions, it is shown that the generating functions are related to the isomonodromic problem of P\(_\text{IV}\). We also show that the formal series for the generating functions are summable. Section 5 is devoted to concluding remarks.

\section{Symmetric Form of P\(_\text{IV}\)}

In this section we give a brief review of the theory of the symmetric form of P\(_\text{IV}\). We refer to \[15\] for details.

\footnote{The convention of composition of transformations is different from what is often used in generating complex exact solutions from simple ones. See the section A.4 of \[15\] for details.}
2.1 Symmetric form and Bäcklund transformations

The symmetric form of $P_{IV}$ (1.1) is given by

\[
\begin{align*}
  f'_0 &= f_0(f_1 - f_2) + \alpha_0, \\
  f'_1 &= f_1(f_2 - f_0) + \alpha_1, \\
  f'_2 &= f_2(f_0 - f_1) + \alpha_2,
\end{align*}
\]  

(2.1)

where $\tau = d/dt$ and

\[
\begin{align*}
  \alpha_0 + \alpha_1 + \alpha_2 &= 1, \\
  f_0 + f_1 + f_2 &= t.
\end{align*}
\]  

(2.2)

We obtain $P_{IV}$ (1.1) for $y = f_2$ by eliminating $f_0$ and $f_1$. The functions $\tau_i$ ($i = 0, 1, 2$) are defined by

\[
\begin{align*}
  h_0 &= \tau_0', \\
  h_1 &= \tau_1', \\
  h_2 &= \tau_2',
\end{align*}
\]  

(2.3)

where $h_i$ ($i = 0, 1, 2$) are Hamiltonians given by

\[
\begin{align*}
  h_0 &= f_0f_1f_2 + \frac{\alpha_1 - \alpha_2}{3}f_0 + \frac{\alpha_1 + 2\alpha_2}{3}f_1 - \frac{2\alpha_1 + \alpha_2}{3}f_2, \\
  h_1 &= f_0f_1f_2 - \frac{2\alpha_2 + \alpha_0}{3}f_0 + \frac{\alpha_2 - \alpha_0}{3}f_1 + \frac{\alpha_2 + 2\alpha_0}{3}f_2, \\
  h_2 &= f_0f_1f_2 + \frac{\alpha_0 + 2\alpha_1}{3}f_0 - \frac{2\alpha_0 + \alpha_1}{3}f_1 + \frac{\alpha_0 - \alpha_1}{3}f_2.
\end{align*}
\]  

(2.4)

The symmetric form of $P_{IV}$ (2.1) admits the following Bäcklund transformations $s_i$ ($i = 0, 1, 2$) and $\pi$ defined by Table 2.1.

<table>
<thead>
<tr>
<th>$s_i$</th>
<th>$\alpha_0$</th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$f_0$</th>
<th>$f_1$</th>
<th>$f_2$</th>
<th>$\tau_0$</th>
<th>$\tau_1$</th>
<th>$\tau_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_0$</td>
<td>$-\alpha_0$</td>
<td>$\alpha_1 + \alpha_0$</td>
<td>$\alpha_2 + \alpha_0$</td>
<td>$f_0$</td>
<td>$f_1 + \frac{\alpha_1}{f_0}$</td>
<td>$f_2 - \frac{\alpha_2}{f_0}$</td>
<td>$f_0\frac{\tau_1}{\tau_0}$</td>
<td>$\tau_1$</td>
<td>$\tau_2$</td>
</tr>
<tr>
<td>$s_1$</td>
<td>$\alpha_0 + \alpha_1$</td>
<td>$-\alpha_1$</td>
<td>$\alpha_2 + \alpha_1$</td>
<td>$f_0 - \frac{\alpha_1}{f_0}$</td>
<td>$f_1$</td>
<td>$f_2 + \frac{\alpha_2}{f_0}$</td>
<td>$\tau_0$</td>
<td>$f_1\frac{\tau_2}{\tau_1}$</td>
<td>$\tau_2$</td>
</tr>
<tr>
<td>$s_2$</td>
<td>$\alpha_0 + \alpha_2$</td>
<td>$\alpha_1 + \alpha_2$</td>
<td>$-\alpha_2$</td>
<td>$f_0 + \frac{\alpha_2}{f_0}$</td>
<td>$f_1 - \frac{\alpha_2}{f_0}$</td>
<td>$f_2$</td>
<td>$\tau_0$</td>
<td>$\tau_1$</td>
<td>$f_2\frac{\tau_0}{\tau_2}$</td>
</tr>
<tr>
<td>$\pi$</td>
<td>$\alpha_1$</td>
<td>$\alpha_2$</td>
<td>$\alpha_0$</td>
<td>$f_1$</td>
<td>$f_2$</td>
<td>$f_0$</td>
<td>$\tau_1$</td>
<td>$\tau_2$</td>
<td>$\tau_0$</td>
</tr>
</tbody>
</table>

Table 2.1: Table of Bäcklund transformations for $P_{IV}$.

Theorem 2.1 (i) $s_i$ ($i = 0, 1, 2$) and $\pi$ commute with derivation.

(ii) $s_i$ ($i = 0, 1, 2$) and $\pi$ satisfy the following fundamental relations

\[
s_i^2 = 1, \quad (s_is_{i+1})^3 = 1, \quad \pi^3 = 1, \quad \pi s_i = s_{i+1}\pi, \quad i \in \mathbb{Z}/3\mathbb{Z},
\]  

(2.5)

and thus $(s_0, s_1, s_2, \pi)$ form the extended affine Weyl group of type $A_2^{(1)}$. 

3
2.2 Bilinear equations for $\tau$ functions

The $f$-variables and the $\tau$ functions are related by

$$
\begin{align*}
    f_0 &= \frac{\tau'_2}{\tau_2} - \frac{\tau'_1}{\tau_1} + \frac{t}{3} = \frac{s_0(\tau_0)\tau_0}{\tau_2\tau_1}, \\
    f_1 &= \frac{\tau'_0}{\tau_0} - \frac{\tau'_2}{\tau_2} + \frac{t}{3} = \frac{s_1(\tau_1)\tau_1}{\tau_0\tau_2}, \\
    f_2 &= \frac{\tau'_1}{\tau_1} - \frac{\tau'_0}{\tau_0} + \frac{t}{3} = \frac{s_2(\tau_2)\tau_2}{\tau_1\tau_0}.
\end{align*}
$$

(2.6)

Moreover, it is shown that $\tau$ functions satisfy various bilinear differential equations of Hirota type. For example, we have:

(I)

$$
\begin{align*}
    (D_t + \frac{t}{3})\tau_2 \cdot \tau_1 &= s_0(\tau_0)\tau_0, \\
    (D_t + \frac{t}{3})\tau_0 \cdot \tau_2 &= s_1(\tau_1)\tau_1, \\
    (D_t + \frac{t}{3})\tau_1 \cdot \tau_0 &= s_2(\tau_2)\tau_2.
\end{align*}
$$

(2.7), (2.8), (2.9)

(II)

$$
\begin{align*}
    \left( D_t^2 + \frac{t}{3}D_t - \frac{2}{9}t^2 + \frac{\alpha_0 - \alpha_1}{3} \right) \tau_0 \cdot \tau_1 &= 0, \\
    \left( D_t^2 + \frac{t}{3}D_t - \frac{2}{9}t^2 + \frac{\alpha_1 - \alpha_2}{3} \right) \tau_1 \cdot \tau_2 &= 0, \\
    \left( D_t^2 + \frac{t}{3}D_t - \frac{2}{9}t^2 + \frac{\alpha_2 - \alpha_0}{3} \right) \tau_2 \cdot \tau_0 &= 0.
\end{align*}
$$

(2.10), (2.11), (2.12)

(III)

$$
\begin{align*}
    \left( \frac{1}{2}D_t^2 - \frac{\alpha_1 - \alpha_2}{3} \right) \tau_0 \cdot \tau_0 &= s_1(\tau_1)s_2(\tau_2), \\
    \left( \frac{1}{2}D_t^2 - \frac{\alpha_2 - \alpha_0}{3} \right) \tau_1 \cdot \tau_1 &= s_2(\tau_2)s_0(\tau_0), \\
    \left( \frac{1}{2}D_t^2 - \frac{\alpha_0 - \alpha_1}{3} \right) \tau_2 \cdot \tau_2 &= s_0(\tau_0)s_1(\tau_1).
\end{align*}
$$

(2.13), (2.14), (2.15)

where $D^n_t$ is the Hirota derivative defined by

$$
D^n_t f \cdot g = \left( \frac{d}{dt} - \frac{d}{ds} \right)^n f(t)g(s) \bigg|_{s=t}.
$$

(2.16)
2.3 Translations and \( \tau \) functions on lattice

Define the translation operators \( T_i \) \((i = 1, 2, 3)\) by

\[
T_1 = \pi s_2 s_1, \quad T_2 = \pi T_1 \pi^{-1} = s_1 \pi s_2, \quad T_3 = \pi T_2 \pi^{-1} = s_2 s_1 \pi. \tag{2.17}
\]

Then it follows that

\[
T_i T_j = T_j T_i \quad (i \neq j), \quad T_1 T_2 T_3 = 1. \tag{2.18}
\]

Actions of \( T_i \) on parameters are given by

\[
T_1(\alpha_0) = \alpha_0 + 1, \quad T_1(\alpha_1) = \alpha_1 - 1, \quad T_1(\alpha_2) = \alpha_2, \tag{2.19}
\]

\[
T_2(\alpha_0) = \alpha_0, \quad T_2(\alpha_1) = \alpha_1 + 1, \quad T_2(\alpha_2) = \alpha_2 - 1, \tag{2.19}
\]

\[
T_3(\alpha_0) = \alpha_0 - 1, \quad T_3(\alpha_1) = \alpha_1, \quad T_3(\alpha_2) = \alpha_2 + 1. \tag{2.19}
\]

We define

\[
\tau_{l,m,n} = T_1^l T_2^m T_3^n (\tau_0), \tag{2.20}
\]

so that

\[
\tau_0 = \tau_{0,0,0}, \quad \tau_1 = \tau_{1,0,0}, \quad \tau_2 = \tau_{0,0,-1}. \tag{2.21}
\]

We note that \( \tau_{l+k,m+k,n+k} = \tau_{l,m,n} \) follows from \( T_1 T_2 T_3 = 1 \). Applying \( T_1^l T_2^m T_3^n \) on the bilinear equations (2.7)-(2.15), we obtain the following equations:

(I)

\[
\left( D_i + \frac{t}{3} \right) \tau_{l+1,m+1,n} \cdot \tau_{l+1,m,n} = \tau_{l+1,m,n-1} \tau_{l+1,m,n}, \tag{2.22}
\]

\[
\left( D_i + \frac{t}{3} \right) \tau_{l,m+1,n} \cdot \tau_{l+1,m+1,n} = \tau_{l,m+1,n-1} \tau_{l+1,m+1,n}, \tag{2.23}
\]

\[
\left( D_i + \frac{t}{3} \right) \tau_{l,m,n} \cdot \tau_{l+1,m,n} = \tau_{l,m-1,n} \tau_{l+1,m+1,n}. \tag{2.24}
\]

(II)

\[
\left( D_i^2 + \frac{t}{3} D_i - \frac{2}{9} t^2 + \frac{\alpha_0 - \alpha_1 + 2l - m - n}{3} \right) \tau_{l,m,n} \cdot \tau_{l+1,m,n} = 0, \tag{2.25}
\]

\[
\left( D_i^2 + \frac{t}{3} D_i - \frac{2}{9} t^2 + \frac{\alpha_1 - \alpha_2 - l + 2m - n}{3} \right) \tau_{l+1,m,n} \cdot \tau_{l+1,m+1,n} = 0, \tag{2.26}
\]

\[
\left( D_i^2 + \frac{t}{3} D_i - \frac{2}{9} t^2 + \frac{\alpha_2 - \alpha_0 - l - m + 2n}{3} \right) \tau_{l+1,m+1,n} \cdot \tau_{l,m,n} = 0. \tag{2.27}
\]

(III)

\[
\left( \frac{1}{2} D_i^2 - \frac{\alpha_1 - \alpha_2 - l + 2m - n}{3} \right) \tau_{l,m,n} \cdot \tau_{l,m,n} = \tau_{l+1,m,n} \tau_{l,m-1,n}, \tag{2.28}
\]

\[
\left( \frac{1}{2} D_i^2 - \frac{\alpha_2 - \alpha_0 - l - m + 2n}{3} \right) \tau_{l+1,m,n} \cdot \tau_{l+1,m,n} = \tau_{l+1,m,n+1} \tau_{l+1,m,n-1}, \tag{2.29}
\]

\[
\left( \frac{1}{2} D_i^2 - \frac{\alpha_0 - \alpha_1 + 2l - m - n}{3} \right) \tau_{l+1,m+1,n} \cdot \tau_{l+1,m+1,n} = \tau_{l+2,m,n+1} \tau_{l+1,m+1,n}. \tag{2.30}
\]
Remark 2.2 Suppose $\tau_0 = \tau_{0,0,0}$, $\tau_1 = \tau_{1,0,0}$ and $\tau_2 = \tau_{1,1,0}$ satisfy the bilinear equations (2.10)-(2.12). Then, $f_0$, $f_1$ and $f_2$ defined by Equations (2.6) satisfy the symmetric form of $P_{IV}$ (2.1). This can be verified as follows. Dividing Equations (2.10) and (2.11) by $\tau_0\tau_1$ and \tau \tau_2, respectively, we have

\[
(h_0 + h_1)' + (h_0 - h_1)^2 + \frac{t}{3}(h_0 - h_1) - \frac{2}{9}t^2 + \frac{\alpha_0 - \alpha_1}{3} = 0,
\]

\[
(h_1 + h_2)' + (h_1 - h_2)^2 + \frac{t}{3}(h_1 - h_2) - \frac{2}{9}t^2 + \frac{\alpha_1 - \alpha_2}{3} = 0.
\]

Subtracting the second equation from the first equation we have

\[
0 = (h_0 - h_2)' + (h_0 - h_2)(h_0 - 2h_1 + h_2) + \frac{t}{3}(h_0 - 2h_1 + h_2) + \frac{1}{3} - \alpha_1
\]

\[
= \left( h_0 - h_2 + \frac{t}{3} \right)' + \left( h_0 - h_2 + \frac{t}{3} \right)(h_0 - 2h_1 + h_2) - \alpha_1 = 0
\]

which is the first equation in Equation (2.1). Here we have used the relations

\[
h_0 - h_2 = f_1 - \frac{t}{3}, \quad h_1 - h_0 = f_2 - \frac{t}{3}, \quad h_2 - h_1 = f_2 - \frac{t}{3},
\]

which follow from Equation (2.4). Other equations in Equation (2.1) can be derived in a similar manner.

Remark 2.3 Applying $T^{l'}_1T^{m'}_2T^{n'}_3$ on Equation (2.6), we have

\[
T^{l'}_1T^{m'}_2T^{n'}_3(f_0) = \frac{\tau_{l+1,m+1,n}'}{\tau_{l+1,m+1,n}} - \frac{\tau_{l+1,m,n}'}{\tau_{l+1,m,n}} + \frac{t}{3} = \frac{\tau_{l+2,m+1,n}T^{l,m,n}}{\tau_{l+1,m+1,n}T^{l+1,m+1,n}},
\]

\[
T^{l'}_1T^{m'}_2T^{n'}_3(f_1) = \frac{\tau_{l+1,m,n}'}{\tau_{l+1,m,n}} - \frac{\tau_{l+1,m+1,n}'}{\tau_{l+1,m+1,n}} + \frac{t}{3} = \frac{\tau_{l+1,m+1,n}T_{l+1,m+1,n}}{\tau_{l+1,m,n}T_{l+1,m+1,n}},
\]

\[
T^{l'}_1T^{m'}_2T^{n'}_3(f_2) = \frac{\tau_{l+1,m+1,n}'}{\tau_{l+1,m+1,n}} - \frac{\tau_{l+1,m,n}'}{\tau_{l+1,m,n}} + \frac{t}{3} = \frac{\tau_{l+1,m,n}T_{l+1,m,n}}{\tau_{l+1,m+1,n}T_{l+1,m+1,n}}.
\]

3 Hankel Determinant Formula

Now consider the sequence of $\tau$ functions $\tau_{n,0,0}$ ($n \in \mathbb{Z}$), which are $\tau$ functions in the direction of $T_1$ on the line $\alpha_2 = \text{const.}$ in the parameter space. It is possible to regard this sequence as being generated by the Toda equation

\[
\left( \frac{1}{2}D^2 - \frac{\alpha_0 - \alpha_1 + 2n - 1}{3} \right)\tau_{n,0,0} = \tau_{n+1,0,0} \tau_{n-1,0,0},
\]

\[ (3.1) \]
from $\tau_0 = \tau_{0,0,0}$ and $\tau_1 = \tau_{1,0,0}$. We note that Equation (3.1) follows from a specialization of Equation (2.30). Let us introduce the variables $\kappa_n$ ($n \in \mathbb{Z}$) by

$$\kappa_n = c^{1/n^2} \frac{\tau_{n,0,0}}{\tau_{0,0,0}},$$

and put

$$\kappa_{-1} = \psi_{-1}, \quad \kappa_1 = \psi_1,$$

where $\psi_{\pm j} = \psi_{\pm j}(t)$. Then, Equation (3.1) can be rewritten as

$$\frac{1}{2} D^2_t \kappa_n = \kappa_{n+1} \kappa_{n-1} - \psi_{-1} \psi_1 \kappa^2_n, \quad \kappa_{-1} = \psi_{-1}, \quad \kappa_0 = 1, \quad \kappa_1 = \psi_1,$$

by using the identities

$$D_t(ab) \cdot (cb) = b^2 D_t a \cdot c,$$

$$D^2_t(ab) \cdot (cb) = (D^2_t a \cdot c) b^2 + ac(D^2_t b \cdot b),$$

and Equation (3.1) with $n = 0$. It is known that $\kappa_n$ can be expressed by a Hankel determinant as follows[11]:

**Theorem 3.1** $\kappa_n$ is given by

$$\kappa_n = \begin{cases} \det(a_{i+j-2})_{1 \leq i,j \leq n} & n > 0 \\ 1 & n = 0 \\ \det(b_{i+j-2})_{1 \leq i,j \leq n} & n < 0 \end{cases}$$

where the entries are defined by the recurrence relations,

$$a_n = a'_{n-1} + \psi_1 \sum_{k=0}^{n-2} a_k a_{n-2-k}, \quad a_0 = \psi_1,$$

$$b_n = b'_{n-1} + \psi_1 \sum_{k=0}^{n-2} b_k b_{n-2-k}, \quad b_0 = \psi_{-1}.$$ 

We note that

$$y_{-1} = -\frac{\psi'_{-1}}{\psi_{-1}} + t, \quad y_0 = \frac{\psi'_1}{\psi_1} + t,$$

satisfy $P_{IV}[\alpha_0 - 1, \alpha_1 + 1, \alpha_2]$ and $P_{IV}[\alpha_0, \alpha_1, \alpha_2]$, respectively. Moreover,

$$y_n = \frac{\kappa'_{n+1}}{\kappa_{n+1}} - \frac{\kappa'_n}{\kappa_n} + t,$$

satisfies $P_{IV}[\alpha_0 + n, \alpha_1 - n, \alpha_2]$,

$$\frac{d^2 y_n}{dt^2} = \frac{1}{2y_n} \left( \frac{dy_n}{dt} \right)^2 + \frac{3}{2} y_n^3 - 2t y_n^2 + \left( \frac{t^2}{2} - \alpha_1 + \alpha_0 + 2n \right) y_n - \frac{\alpha_2^2}{2y_n}.$$ 

We also note that the determinant formula for the $\tau$ sequences in the directions of $T_2$ and $T_3$ are formulated in a similar manner.
4 Generating Functions and Isomonodromic Problem

4.1 Riccati Equations

For the determinant formula Theorem 3.1, let us consider the generating functions of the entries

\[ F_1(t, \lambda) = \sum_{n=0}^{\infty} a_n \lambda^{-n}, \quad G_1(t, \lambda) = \sum_{n=0}^{\infty} b_n \lambda^{-n}, \]

(4.1)

where \(a_n\) and \(b_n\) are characterized by the recurrence relations (3.6) and (3.7). Multiplying Equations (3.6) and (3.7) by \(n\), and taking summation over \(n\), we obtain the following Riccati equations for the generating functions \(F_1\) and \(G_1\).

**Proposition 4.1** \(F_1(t, \lambda)\) and \(G_1(t, \lambda)\) satisfy the Riccati equations

\[ \lambda \frac{\partial F}{\partial t} = -\psi_1 F^2 + \lambda^2 F - \lambda^2 \psi_1, \]

(4.2)

\[ \lambda \frac{\partial G}{\partial t} = -\psi_1 G^2 + \lambda^2 G - \lambda^2 \psi_1, \]

(4.3)

respectively.

Since \(F_\infty\) and \(G_\infty\) are defined as formal power series at \(\lambda = \infty\), it is convenient to derive differential equations that they satisfy with respect to \(\lambda\). The following auxiliary recurrence relations for \(a_n\) and \(b_n\) are useful for this purpose.

**Lemma 4.2** \(a_n\) and \(b_n\) satisfy

\[ \frac{d}{dt} [\psi_{-1} a_n - (\psi_{-1}' - t \psi_{-1}) a_{n-1}] + n \psi_{-1} a_{n-1} = 0, \]

(4.4)

and

\[ \frac{d}{dt} [\psi_1 b_n - (\psi_1' + t \psi_1) b_{n-1}] - n \psi_1 b_{n-1} = 0, \]

(4.5)

respectively.

The proof of Lemma 4.2 is achieved by tedious but straightforward induction by noticing the relations

\[ \psi_1'' + t \psi_1' + 2 \psi_1^2 \psi_{-1} + (\alpha_0 - \alpha_1) \psi_1 = 0, \]

(4.6)

\[ \psi_{-1}'' - t \psi_{-1}' + 2 \psi_{-1}^2 \psi_1 + (\alpha_0 - \alpha_1 - 2) \psi_{-1} = 0, \]

(4.7)

which follow from the bilinear equation (2.25) with \((l, \ell, n) = (0, 0, 0)\) and \((-1, 0, 0)\), respectively. Lemma 4.2 yields the following linear partial differential equations for \(F_\infty\) and \(G_\infty\):
Lemma 4.3 \( F_\infty(t, \lambda) \) and \( G_\infty(t, \lambda) \) satisfy the linear differential equations

\[
\begin{aligned}
\left( \lambda + t - \frac{\psi'_1}{\psi_{-1}} \right) \frac{\partial F}{\partial t} - \lambda \frac{\partial F}{\partial \lambda} &= - \left( \lambda + t \right) \frac{\psi'_{-1}}{\psi_{-1}} - \frac{\psi''_{-1}}{\psi_{-1}} + 2 \right) F + \frac{\lambda}{\psi_{-1}} (\psi_{-1} \psi'_1)',
\end{aligned}
\] (4.8)

\[
\begin{aligned}
\left( -\lambda + t + \frac{\psi'_1}{\psi_1} \right) \frac{\partial G}{\partial t} - \lambda \frac{\partial G}{\partial \lambda} &= - \left( (-\lambda + t) \frac{\psi'_{1}}{\psi_1} + \frac{\psi''_{1}}{\psi_1} + 2 \right) G - \frac{\lambda}{\psi_1} (\psi_{1} \psi'_1)',
\end{aligned}
\] (4.9)

respectively.

Eliminating \( t \)-derivatives from Equations (4.2) and (4.8), and from Equations (4.3) and (4.9), respectively, we obtain the following Riccati equations with respect to \( \lambda \).

Proposition 4.4 \( F_\infty(t, \lambda) \) and \( G_\infty(t, \lambda) \) satisfy the following Riccati equations

\[
\begin{aligned}
\lambda^2 \frac{\partial F}{\partial \lambda} &= - \left( \lambda + t - \frac{\psi'_1}{\psi_{-1}} \right) \psi_{-1} F^2 \\
&+ \lambda \left( \lambda^2 + \lambda t + 2 \psi_1 \psi_{-1} + \alpha_0 - \alpha_1 \right) F - \lambda^2 \left( (\lambda + t) \psi_1 + \psi'_1 \right),
\end{aligned}
\] (4.10)

\[
\begin{aligned}
\lambda^2 \frac{\partial G}{\partial \lambda} &= - \left( -\lambda + t + \frac{\psi'_1}{\psi_1} \right) \psi_1 G^2 \\
&- \lambda \left( \lambda^2 - \lambda t + 2 \psi_1 \psi_{-1} + \alpha_0 - \alpha_1 - 2 \right) G - \lambda^2 \left( (-\lambda + t) \psi_{-1} - \psi'_{-1} \right),
\end{aligned}
\] (4.11)

respectively.

4.2 Isomonodromic Problem

The Riccati equations in Proposition 4.1 and Proposition 4.4 can be linearized into second order linear differential equations by the standard technique, which yields isomonodromic problems associated with \( P_{IV} \).

Theorem 4.5 (i) It is possible to introduce the functions \( Y_1 \) and \( Y_2 \) consistently as

\[
\begin{aligned}
F_\infty(t, \lambda) &= \frac{\lambda}{\psi_{-1}} \left( \frac{1}{Y_1} \frac{\partial Y_1}{\partial t} + \frac{\lambda}{2} \right) \\
&= \frac{\lambda^2}{\psi_{-1}^2} \left( \frac{1}{Y_1} \frac{\partial Y_1}{\partial t} + \frac{\lambda}{2} \right) \\
&\quad \times \left( \frac{1}{Y_1} \frac{\partial Y_1}{\partial t} + \frac{\lambda + t}{2} + \frac{\psi_{-1} \psi_1 + \alpha_0 - 1}{\lambda} + \frac{\alpha_2}{2 \lambda} \right),
\end{aligned}
\] (4.12)

\[
Y_2 = \frac{1}{\psi_{-1}} \left( \frac{\partial Y_1}{\partial t} + \frac{\lambda}{2} Y_1 \right).
\] (4.13)
Then the Riccati equations (4.2) and (4.10) are linearized to:

$$\frac{\partial}{\partial \lambda} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = A \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \quad \frac{\partial}{\partial t} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = B \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix},$$

(4.14)

where

$$u = \psi_-, \quad y_- = -\frac{\psi'_-}{\psi_-} + t, \quad z = \psi_-\psi_1 + \alpha_0 - 1 = \psi_-\psi_1 - \alpha_1 - \alpha_2. \quad (4.17)$$

Conversely, if $Y_1$ and $Y_2$ are the solutions of linear system (4.14)-(4.16), then

$$F = \frac{Y_2}{Y_1},$$

satisfies the Riccati equations (4.2) and (4.10).

(ii) It is possible to introduce the functions $Z_1$ and $Z_2$ consistently as

$$G_\infty(t, \lambda) = \frac{\lambda}{\psi_1} \left( \frac{1}{Z_1} \frac{\partial Z_1}{\partial t} + \frac{\lambda}{2} \frac{\partial^2 Z_1}{\partial t^2} \right)$$

$$= \frac{\lambda^2}{\psi_1} \frac{1}{t + \psi'_- \psi_1} \times \left( \frac{1}{Z_1} \frac{\partial Z_1}{\partial \lambda} + \frac{-\lambda + t + \psi_-\psi_1 + \alpha_0 - 1 + \frac{\alpha_2}{2\lambda}}{s} \right), \quad (4.19)$$

$$Z_2 = \frac{1}{\psi_1} \left( \frac{\partial Z_1}{\partial t} + \frac{\lambda}{2} Z_1 \right). \quad (4.20)$$

Then the Riccati equations (4.3) and (4.11) are linearized to:

$$\frac{\partial}{\partial \lambda} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = C \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}, \quad \frac{\partial}{\partial t} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = D \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix},$$

(4.21)
\[
C = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \lambda + \begin{pmatrix} -\frac{t}{2} & -\frac{v}{2} \\ \frac{w + \alpha_1 + \alpha_2}{v} & \frac{t}{2} \end{pmatrix} + \begin{pmatrix} \frac{w + \alpha_2}{2} & \frac{vy_0}{2} \\ \frac{w(w + \alpha_2)}{vy_0} & -w - \frac{\alpha_2}{2} \end{pmatrix} \lambda^{-1},
\]
\[\text{and} \quad D = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \lambda + \begin{pmatrix} 0 & 0 \\ -\frac{w + \alpha_1 + \alpha_2}{v} & \frac{v}{2} \end{pmatrix} \] (4.24)

where
\[v = \psi, \quad y_0 = \frac{\psi'}{\psi} + t, \quad w = \psi' - \alpha_0 - 1 = \psi' - \alpha_1 - \alpha_2.\] (4.25)

Conversely, if \(Z_1\) and \(Z_2\) are the solutions of linear system (4.21)-(4.24), then
\[G = \lambda \frac{Z_2}{Z_1},\] (4.26)
satisfies the Riccati equations (4.3) and (4.11).

**Remark 4.6** The linear systems (4.14)-(4.16) and (4.21)-(4.24) are nothing but the isomonodromic problem for \(P_{IV}[\alpha_0 - 1, \alpha_1 + 1, \alpha_2]\) and \(P_{IV}[\alpha_0, \alpha_1, \alpha_2]\), respectively\[7\]. In fact, compatibility condition of the linear system (4.14)-(4.16)
\[
\frac{\partial A}{\partial t} - \frac{\partial B}{\partial \lambda} + AB - BA = 0,
\] (4.27)
gives
\[
\frac{dz}{dt} = \frac{z^2}{y_{-1}} + \left( \frac{\alpha_2}{y_{-1}} - y_{-1} \right) z - (\alpha_1 + \alpha_2)y_{-1},
\] (4.28)
\[
\frac{dy_{-1}}{dt} = 2z + y_{-1}^2 - ty_{-1} + \alpha_2,
\] (4.29)
\[
\frac{du}{dt} = (-y_{-1} + t)u.
\] (4.30)

Eliminating \(z\), we have \(P_{IV}[\alpha_0 - 1, \alpha_1 + 1, \alpha_2]\) for \(y_{-1}\)
\[
y_{-1}'' = \frac{(y_{-1}')^2}{2y_{-1}} + \frac{3}{2} \frac{y_{-1}^3}{y_{-1}} - 2ty_{-1}^2 + \left( \frac{t^2}{2} - \alpha_1 + \alpha_0 - 2 \right)y_{-1} - \frac{\alpha_2^2}{2y_{-1}}.
\] (4.31)

This fact also establishes the consistency of two expressions of \(F_{\infty}(t, \lambda)\) in terms of \(Y_1\) in Equation (4.12). A similar remark holds true for \(G_{\infty}(t, \lambda)\) and \(Z_1\).
Remark 4.7 From Equation (4.12), \( Y_1 \) can be formally expressed as

\[
Y_1 = \text{const.} \times \exp \left( -\frac{\lambda^2}{4} - \frac{\lambda t}{2} \right) \lambda^{\alpha_1 + \alpha_2 / 2} \exp \left( -\sum_{n=1}^{\infty} \lambda^{-n} \int \psi_\gamma(a_{n-1} dt) \right),
\]

which coincides with the known asymptotic behavior of \( Y_1 \) around \( \lambda \sim \infty \).[7]

4.3 Solutions of Isomonodromic Problems and Determinant Formula

We have investigated the generating functions \( F_1 \) and \( G_1 \) of entries of the Hankel determinant formula and shown that they formally satisfy the Riccati equations (4.2), (4.10) and (4.3), (4.11), respectively, and that those Riccati equations are linearized into isomonodromic problems (4.14)-(4.16) and (4.21)-(4.24) for \( P_{IV} \).

Now let us start from the isomonodromic problem (4.14)-(4.16). We have two linearly independent solutions around \( \lambda = \infty \), one of which is related to the generating function \( F_1 \) by

\[
F_1(t) = Y_2(t) = Y_1(t).
\]

So let us consider another solution. It is known that the linear system (4.14)-(4.16) admits the following formal solutions[7] around \( \lambda = \infty \):

\[
\begin{pmatrix}
Y^{(1)}_1 \\
Y^{(1)}_2
\end{pmatrix} = \exp \left( -\frac{\lambda^2}{4} - \frac{\lambda t}{2} \right) \lambda^{\alpha_1 + \alpha_2 / 2} \left[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} y^{(1)}_1 \\ y^{(1)}_2 \end{pmatrix} \lambda^{-1} + \cdots \right],
\]

\[
\begin{pmatrix}
Y^{(2)}_1 \\
Y^{(2)}_2
\end{pmatrix} = \exp \left( \frac{\lambda^2}{4} + \frac{\lambda t}{2} \right) \lambda^{-\alpha_1 - \alpha_2 / 2} \left[ \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} y^{(2)}_1 \\ y^{(2)}_2 \end{pmatrix} \lambda^{-1} + \cdots \right].
\]

These solutions give

\[
F^{(1)}(t, \lambda) = \lambda \frac{Y_2^{(1)}}{Y_1^{(1)}} = \lambda \times \frac{y_2^{(1)} \lambda^{-1} + \cdots}{1 + y_1^{(1)} \lambda^{-1} + \cdots} = a_0 + a_1 \lambda^{-1} + \cdots,
\]

\[
F^{(2)}(t, \lambda) = \lambda \frac{Y_2^{(2)}}{Y_1^{(2)}} = \lambda \times \frac{1 + y_1^{(2)} \lambda^{-1} + \cdots}{y_2^{(2)} \lambda^{-1} + \cdots} = \lambda^2 (c_0 + c_1 \lambda^{-1} + \cdots),
\]

respectively. Theorem 4.5 states that both \( F^{(1)}(t, \lambda) \) and \( F^{(2)}(t, \lambda) \) satisfy the Riccati equations (4.2) and (4.10). Conversely, the above two possibilities of power-series solutions for the Riccati equations are verified directly.

Proposition 4.8 The Riccati equations (4.2) and (4.10) admit only the following two kinds of power-series solutions around \( \lambda \sim \infty \):

\[
F^{(1)}(t, \lambda) = \sum_{n=0}^{\infty} a_n \lambda^{-n}, \quad F^{(2)}(t, \lambda) = \lambda^2 \sum_{n=0}^{\infty} c_n \lambda^{-n}.
\]

(4.33)
The proof of Proposition 4.8 is achieved simply by plugging the power-series solution
\[ F = \lambda^\rho \sum_{n=0}^{\infty} u_n \lambda^{-n}, \]
into the Riccati equations (4.2) and (4.10), and investigating the balance of leading terms. Then we find that \( \rho = 0, 2 \). A similar result can be shown for the Riccati equations (4.3) and (4.11).

**Proposition 4.9** The Riccati equations (4.3) and (4.11) admit only the following two kinds of power-series solutions around \( \lambda = \infty \):
\[ G^{(1)}(t, \lambda) = \sum_{n=0}^{\infty} b_n \lambda^{-n}, \quad G^{(2)}(t, \lambda) = \lambda^2 \sum_{n=0}^{\infty} d_n \lambda^{-n}. \] (4.34)

It is obvious that \( F^{(1)}(t, \lambda) \) and \( G^{(1)}(t, \lambda) \) are nothing but \( F_\infty(t, \lambda) \) and \( G_\infty(t, \lambda) \), respectively. Therefore it is an important problem to investigate \( F^{(2)}(t, \lambda) \) and \( G^{(2)}(t, \lambda) \). Now we present two observations regarding this problem. The first observation is that there are quite simple relations among those functions:

**Proposition 4.10** The following relations hold:
\[ F^{(2)}(t, \lambda) = \frac{\lambda^2}{G^{(1)}(t, -\lambda)}, \quad G^{(2)}(t, \lambda) = \lambda^2 F^{(1)}(t, -\lambda). \] (4.35)

**Proof.** Substituting \( F(t, \lambda) = \lambda^2 / g(t, \lambda) \) into the Riccati equations (4.2) and (4.10), we obtain Equations (4.3) and (4.11), respectively, for \( G(t, \lambda) = g(t, -\lambda) \). Choosing \( g(t, \lambda) = G^{(1)}(t, \lambda) \), \( F(t, \lambda) \) must be \( F^{(2)}(t, \lambda) \), since its leading order is \( \lambda^2 \). We obtain the second equation by the similar argument. \( \square \)

Secondly, \( F^{(2)}(t, \lambda) \) and \( G^{(2)}(t, \lambda) \) can be also interpreted as generating functions of the Hankel determinant formula for \( P_{IV} \). Recall that the determinant formula in Theorem 3.1 is for the \( \tau \) sequence \( \kappa_n = e^{-m^2/3} \tau_{n,0,0}/\tau_{0,0,0} \). The following Proposition states that \( F^{(2)}(t, \lambda) \) and \( G^{(2)}(t, \lambda) \) correspond to different normalizations of the \( \tau \) sequence:

**Proposition 4.11** Let
\[ F^{(2)}(t, \lambda) = -\frac{\lambda^2}{\psi_{\lambda}^{\frac{1}{3}}} \sum_{n=0}^{\infty} c_n(-\lambda)^{-n}, \] (4.36)
\[ G^{(2)}(t, \lambda) = -\frac{\lambda^2}{\psi_{\lambda}^{\frac{1}{3}}} \sum_{n=0}^{\infty} d_n(-\lambda)^{-n}, \] (4.37)
be formal solutions of the Riccati equations (4.2), (4.10) and (4.3), (4.11), respectively. Then we have the following:

13
(i) $c_0 = -\psi_{-1}$ and $c_1 = \psi'_{-1}$. For $n \geq 2$, $c_n$’s are characterized by the recursion relation
\[
c_{n+1} = c_n' + \frac{1}{\psi_{-1}} \sum_{k=2}^{n-1} c_k c_{n+1-k}, \quad c_2 = \frac{\psi''_{-1} \psi_{-1} - (\psi'_{-1})^2 + \psi^3_{-1} \psi_1}{\psi_{-1}}. \tag{4.38}
\]

(ii) $d_0 = -\psi_1$ and $d_1 = \psi'_1$. For $n \geq 2$, $d_n$’s are characterized by the recursion relation
\[
d_{n+1} = d_n' + \frac{1}{\psi_1} \sum_{k=2}^{n-1} d_k d_{n+1-k}, \quad d_2 = \frac{\psi''_1 \psi_1 - (\psi'_1)^2 + \psi^{-1}_1}{\psi_1}. \tag{4.39}
\]

(iii) We put
\[
\sigma_{-n-1} = \det(c_{i+j})_{i,j=1,\ldots,n} \quad (n > 0), \quad \sigma_{-1} = 1, \quad \theta_{n+1} = \det(d_{i+j})_{i,j=1,\ldots,n} \quad (n > 0), \quad \theta_1 = 1. \tag{4.40, 4.41}
\]

Then $\sigma_n$ and $\theta_n$ are related to $\tau_{n,0,0}$ as
\[
\sigma_n = \frac{\kappa_n}{\kappa_{-1}} = e^{-\frac{1}{2}(n+1)^2} \frac{\tau_{n,0,0}}{\tau_{-1,0,0}} \quad (n < 0), \tag{4.42}
\]
\[
\theta_n = \frac{\kappa_n}{\kappa_1} = e^{-\frac{1}{2}(n-1)^2} \frac{\tau_{n,0,0}}{\tau_{1,0,0}} \quad (n > 0). \tag{4.43}
\]

Proof. (i) and (ii) can be proved easily by substituting Equations (4.36) and (4.37) into the Riccati equations (4.2) and (4.3), respectively, and collecting the coefficients of powers of $\lambda$. For (iii), we notice that from the Toda equation (3.4) with $n = -1$, namely
\[
\psi''_{-1} \psi_{-1} - (\psi'_{-1})^2 = \kappa_{-2} - \psi^3_{-1} \psi_1,
\]
we have
\[
c_2 = \frac{\psi''_{-1} \psi_{-1} - (\psi'_{-1})^2 + \psi^3_{-1} \psi_1}{\psi_{-1}} = \frac{\kappa_{-2}}{\kappa_{-1}} = \sigma_{-2}.
\]

Moreover, the coefficient of the quadratic term in Equation (4.38) can be regarded as
\[
\frac{1}{\psi_{-1}} = \frac{\kappa_0}{\kappa_{-1}} = \sigma_0.
\]

Applying Theorem 3.1, we find that $\sigma_{-n-1} = \det(c_{i+j})_{i,j=1,\ldots,n} (n > 0)$. The statement for $d_n$ can be shown by a similar argument. □
4.4 Summability of the Generating Functions

To study the growth as $n \to \infty$ of the coefficients $a_n(t)$ (or $b_n(t)$) in Equation (4.1), we use a theorem proved in [5].

**Theorem 4.12 (Hsieh and Sibuya [5], Theorem XIII-8-3)** Consider the following non-linear differential equation in the variable $s$

$$s^{k+1} \frac{dH}{ds} = c(s)H + s b(s, H)$$

(4.44)

where $k$ is a positive integer, $c(s)$ is holomorphic in the neighbourhood of $s = 0$ and $c(0) \neq 0$, and $b(s, H)$ is holomorphic in the neighbourhood of $(s, H) = (0, 0)$. Then equation (4.44) admits one and only one formal solution $H_f(s)$ of the form $H_f(s) = \sum_{n=1}^{\infty} a_n s^n$. Moreover, $H_f$ is $k$-summable in any direction $\arg(s) = \vartheta$ except a finite number of values $\vartheta$. Furthermore, the sum of $H_f(s)$ in the direction $\arg(s) = \vartheta$ is a solution of Equation (4.44).

Equation (4.10) can be put into the form (4.44) by changing variables to $t = 1/s$ and taking $H = F - a_0 = F - \psi_1$. We then obtain Equation (4.44) with $k = 2$, $c(s) \equiv -1$ and

$$b(s, H) = -t H + \psi'_1 + s(\psi_{-1} H^2 - (\alpha_0 - \alpha_1)(H + \psi_1)) + s^2(t\psi_{-1} - \psi'_1)(H + \psi_1)^2$$

Applying Theorem 4.12, we deduce that Equation (4.10) admits one and only one formal solution $F_\infty(\lambda)$ of the form $\sum_{n=0}^{\infty} a_n \lambda^n$. This formal solution is 2-summable in any direction $\arg(\lambda) = \vartheta$ except a finite number of values $\vartheta$ and its sum in the direction $\arg(\lambda) = \vartheta$ is a solution of Equation (4.10).

The definition of $k$-summability implies that $F_\infty(\lambda)$ is of Gevrey order 2, namely, for each $t$, there exist positive numbers $C(t)$ and $K(t)$ such that

$$|a_n(t)| < C(t)(n!)^2 K(t)^n, \quad \text{for all } n \geq 1.$$

Clearly, one can prove a similar result for the coefficients $b_n$ of the formal solutions $G_\infty$ in Equation (4.1), as well as the coefficients $c_n, d_n$ of the formal series in Equations (4.33) and (4.34). For the formal solutions $F^{(2)}(t, \lambda)$ (or $G^{(2)}(t, \lambda)$), we need to apply Theorem 4.12 to a new $H(s) = s^2 F^{(2)} - c_0$ (or $s^2 G^{(2)} - d_0$).

We also note that summability of those series implies that they are expressible in terms of inverse Laplace transformation of certain analytic functions. For details, we refer to [2].

5 Concluding Remarks

In this article, we have constructed a Hankel determinant formula for the $\tau$ sequence of $P_{1V}$ in the direction of translation $T_1$. Then we have shown that the generating functions of
the entries are closely related to the solutions of isomonodromic problems. More precisely, coefficients of asymptotic expansion of the ratio of solutions for isomonodromic problem give the entries of Hankel determinant formula. Moreover, we have shown that there exist simple but mysterious relations among those generating functions. We also discussed the summability of the generating functions.

Let us finally give some remarks. Firstly, in this article we have considered only the $\tau$ sequences in the direction of $T_1$. It is also possible to consider the directions $T_2$, $T_3$ by the $\tilde{ \mathcal{W}}(A_{2}^{(1)})$ symmetry of $P_{IV}$. Secondly, it is surprising that the results obtained in this article are completely parallel to the $P_{II}$ case [8], although concrete computations depend on the specific situation for each case. In particular, it is remarkable that many formulas have exactly the same form as the $P_{II}$ case, such as Equation (4.32) (except for the dominant exponential factor), or formulas in Section 4.3. This coincidence may imply that (i) the phenomena observed for $P_{II}$ and $P_{IV}$ could be universal; at least it can be seen for other Painlevé equations, (ii) the underlying mathematical structure may originate from the Toda equation, rather than the Painlevé equations themselves.

These points will be explored in forthcoming articles.

**Acknowledgments** Nalini Joshi’s research is supported by the Australian Research Council Discovery Project Grant #DP0345505. K. Kajiwara is partially supported by the JSPS Grant-in-Aid for Scientific Research (B)15340057 and (A)16204007. He acknowledges the support by the 21st Century COE program “Development of Dynamic Mathematics with High Functionality” at the Faculty of Mathematics, Kyushu University. He also thank warm hospitality of the School of Mathematics and Statistics of the University of Sydney, where the most of the results have been obtained during his stay. The researches by M. Mazzocco were partially supported by EPSRC and by the Marie Curie Research Training Network ENIGMA.

**References**


List of MHF Preprint Series, Kyushu University
21st Century COE Program
Development of Dynamic Mathematics with High Functionality

MHF2003-1 Mitsuhiro T. NAKAO, Kouji HASHIMOTO & Yoshitaka WATANABE
A numerical method to verify the invertibility of linear elliptic operators with applications to nonlinear problems

MHF2003-2 Masahisa TABATA & Daisuke TAGAMI
Error estimates of finite element methods for nonstationary thermal convection problems with temperature-dependent coefficients

MHF2003-3 Tomohiro ANDO, Sadanori KONISHI & Seiya IMOTO
Adaptive learning machines for nonlinear classification and Bayesian information criteria

MHF2003-4 Kazuhiro YOKOYAMA
On systems of algebraic equations with parametric exponents

MHF2003-5 Masao ISHIKAWA & Masato WAKAYAMA
Applications of Minor Summation Formulas III, Plücker relations, Lattice paths and Pfaffian identities

MHF2003-6 Atsushi SUZUKI & Masahisa TABATA
Finite element matrices in congruent subdomains and their effective use for large-scale computations

MHF2003-7 Setsuo TANIGUCHI
Stochastic oscillatory integrals - asymptotic and exact expressions for quadratic phase functions -

MHF2003-8 Shoki MIYAMOTO & Atsushi YOSHIKAWA
Computable sequences in the Sobolev spaces

MHF2003-9 Toru FUJII & Takashi YANAGAWA
Wavelet based estimate for non-linear and non-stationary auto-regressive model

MHF2003-10 Atsushi YOSHIKAWA
Maple and wave-front tracking — an experiment

MHF2003-11 Masanobu KANEKO
On the local factor of the zeta function of quadratic orders

MHF2003-12 Hidefumi KAWASAKI
Conjugate-set game for a nonlinear programming problem
MHF2004-1 Koji YONEMOTO & Takashi YANAGAWA
Estimating the Lyapunov exponent from chaotic time series with dynamic noise

MHF2004-2 Rui YAMAGUCHI, Eiko TSUCHIYA & Tomoyuki HIGUCHI
State space modeling approach to decompose daily sales of a restaurant into time-dependent multi-factors

MHF2004-3 Kenji KAJIWARA, Tetsu MASUDA, Masatoshi NOUMI, Yasuhiro OHTA & Yasuhiko YAMADA
Cubic pencils and Painlevé Hamiltonians

MHF2004-4 Atsushi KAWAGUCHI, Koji YONEMOTO & Takashi YANAGAWA
Estimating the correlation dimension from a chaotic system with dynamic noise

MHF2004-5 Atsushi KAWAGUCHI, Kentarou KITAMURA, Koji YONEMOTO, Takashi YANAGAWA & Kiyofumi YUMOTO
Detection of auroral breakups using the correlation dimension

MHF2004-6 Ryo IKOTA, Masayasu MIMURA & Tatsuyuki NAKAKI
A methodology for numerical simulations to a singular limit

MHF2004-7 Ryo IKOTA & Eiji YANAGIDA
Stability of stationary interfaces of binary-tree type

MHF2004-8 Yuko ARAKI, Sadanori KONISHI & Seiya IMOTO
Functional discriminant analysis for gene expression data via radial basis expansion

MHF2004-9 Kenji KAJIWARA, Tetsu MASUDA, Masatoshi NOUMI, Yasuhiro OHTA & Yasuhiko YAMADA
Hypergeometric solutions to the q-Painlevé equations

MHF2004-10 Raimundas VIDŪNAS
Expressions for values of the gamma function

MHF2004-11 Raimundas VIDŪNAS
Transformations of Gauss hypergeometric functions

MHF2004-12 Koji NAKAGAWA & Masakazu SUZUKI
Mathematical knowledge browser

MHF2004-13 Ken-ichi MARUNO, Wen-Xiu MA & Masayuki OIKAWA
Generalized Casorati determinant and Positon-Negaton-Type solutions of the Toda lattice equation

MHF2004-14 Nalini JOSHI, Kenji KAJIWARA & Marta MAZZOCCH
Generating function associated with the determinant formula for the solutions of the Painlevé II equation
MHF2004-15 Kouji HASHIMOTO, Ryohei ABE, Mitsuhiro T. NAKAO & Yoshitaka WATANABE
Numerical verification methods of solutions for nonlinear singularly perturbed problem

MHF2004-16 Ken-ichi MARUNO & Gino BIONDINI
Resonance and web structure in discrete soliton systems: the two-dimensional Toda lattice and its fully discrete and ultra-discrete versions

MHF2004-17 Ryuei NISHII & Shinto EGUCHI
Supervised image classification in Markov random field models with Jeffreys divergence

MHF2004-18 Kouji HASHIMOTO, Kenta KOBAYASHI & Mitsuhiro T. NAKAO
Numerical verification methods of solutions for the free boundary problem

MHF2004-19 Hiroki MASUDA
Ergodicity and exponential $\beta$-mixing bounds for a strong solution of Lévy-driven stochastic differential equations

MHF2004-20 Setsuo TANIGUCHI
The Brownian sheet and the reflectionless potentials

MHF2004-21 Ryuei NISHII & Shinto EGUCHI
Supervised image classification based on AdaBoost with contextual weak classifiers

MHF2004-22 Hideki KOSAKI
On intersections of domains of unbounded positive operators

MHF2004-23 Masahisa TABATA & Shoichi FUJIMA
Robustness of a characteristic finite element scheme of second order in time increment

MHF2004-24 Ken-ichi MARUNO, Adrian ANKIEWICZ & Nail AKHMEDEIEV
Dissipative solitons of the discrete complex cubic-quintic Ginzburg-Landau equation

MHF2004-25 Raimundas VIDŪNAS
Degenerate Gauss hypergeometric functions

MHF2004-26 Ryo IKOTA
The boundedness of propagation speeds of disturbances for reaction-diffusion systems

MHF2004-27 Ryusuke KON
Convex dominates concave: an exclusion principle in discrete-time Kolmogorov systems
MHF2004-28 Ryusuke KON
Multiple attractors in host-parasitoid interactions: coexistence and extinction

MHF2004-29 Kentaro IHARA, Masanobu KANEKO & Don ZAGIER
Derivation and double shuffle relations for multiple zeta values

MHF2004-30 Shuichi INOKUCHI & Yoshihiro MIZOGUCHI
Generalized partitioned quantum cellular automata and quantization of classical CA

MHF2005-1 Hideki KOSAKI
Matrix trace inequalities related to uncertainty principle

MHF2005-2 Masahisa TABATA
Discrepancy between theory and real computation on the stability of some finite element schemes

MHF2005-3 Yuko ARAKI & Sadanori KONISHI
Functional regression modeling via regularized basis expansions and model selection

MHF2005-4 Yuko ARAKI & Sadanori KONISHI
Functional discriminant analysis via regularized basis expansions

MHF2005-5 Kenji KAJIWARA, Tetsu MASUDA, Masatoshi NOUMI, Yasuhiro OHTA & Yasuhiko YAMADA
Point configurations, Cremona transformations and the elliptic difference Painlevé equations

MHF2005-6 Kenji KAJIWARA, Tetsu MASUDA, Masatoshi NOUMI, Yasuhiro OHTA & Yasuhiko YAMADA
Construction of hypergeometric solutions to the $q$-Painlevé equations

MHF2005-7 Hiroki MASUDA
Simple estimators for non-linear Markovian trend from sampled data: I. ergodic cases

MHF2005-8 Hiroki MASUDA & Nakahiro YOSHIDA
Edgeworth expansion for a class of Ornstein-Uhlenbeck-based models

MHF2005-9 Masayuki UCHIDA
Approximate martingale estimating functions under small perturbations of dynamical systems

MHF2005-10 Ryo MATSUZAKI & Masayuki UCHIDA
One-step estimators for diffusion processes with small dispersion parameters from discrete observations

MHF2005-11 Junichi MATSKUKUBO, Ryo MATSUZAKI & Masayuki UCHIDA
Estimation for a discretely observed small diffusion process with a linear drift
MHF2005-12 Masayuki UCHIDA & Nakahiro YOSHIDA
AIC for ergodic diffusion processes from discrete observations

MHF2005-13 Hiromichi GOTO & Kenji KAJIWARA
Generating function related to the Okamoto polynomials for the Painlevé IV equation

MHF2005-14 Masato KIMURA & Shin-ichi NAGATA
Precise asymptotic behaviour of the first eigenvalue of Sturm-Liouville problems with large drift

MHF2005-15 Daisuke TAGAMI & Masahisa TABATA
Numerical computations of a melting glass convection in the furnace

MHF2005-16 Raimundas VIDŪNAS
Normalized Leonard pairs and Askey-Wilson relations

MHF2005-17 Raimundas VIDŪNAS
Askey-Wilson relations and Leonard pairs

MHF2005-18 Kenji KAJIWARA & Atsushi MUKAIHIRA
Soliton solutions for the non-autonomous discrete-time Toda lattice equation

MHF2005-19 Yuu HARIYA
Construction of Gibbs measures for 1-dimensional continuum fields

MHF2005-20 Yuu HARIYA
Integration by parts formulae for the Wiener measure restricted to subsets in \( \mathbb{R}^d \)

MHF2005-21 Yuu HARIYA
A time-change approach to Kotani’s extension of Yor’s formula

MHF2005-22 Tadahisa FUNAKI, Yuu HARIYA & Mark YOR
Wiener integrals for centered powers of Bessel processes, I

MHF2005-23 Masahisa TABATA & Satoshi KAIZU
Finite element schemes for two-fluids flow problems

MHF2005-24 Ken-ichi MARUNO & Yasuhiro OHTA
Determinant form of dark soliton solutions of the discrete nonlinear Schrödinger equation

MHF2005-25 Alexander V. KITAEV & Raimundas VIDŪNAS
Quadratic transformations of the sixth Painlevé equation

MHF2005-26 Toru FUJII & Sadanori KONISHI
Nonlinear regression modeling via regularized wavelets and smoothing parameter selection
MHF2005-27 Shuichi INOKUCHI, Kazumasa HONDA, Hyen Yeal LEE, Tatsuro SATO, Yoshihiro MIZOGUCHI & Yasuo KAWAHARA
On reversible cellular automata with finite cell array

MHF2005-28 Toru KOMATSU
Cyclic cubic field with explicit Artin symbols

MHF2005-29 Mitsuhiro T. NAKAO, Kouji HASHIMOTO & Kaori NAGATOU
A computational approach to constructive a priori and a posteriori error estimates for finite element approximations of bi-harmonic problems

MHF2005-30 Kaori NAGATOU, Kouji HASHIMOTO & Mitsuhiro T. NAKAO
Numerical verification of stationary solutions for Navier-Stokes problems

MHF2005-31 Hidefumi KAWASAKI
A duality theorem for a three-phase partition problem

MHF2005-32 Hidefumi KAWASAKI
A duality theorem based on triangles separating three convex sets

MHF2005-33 Takeaki FUCHIKAMI & Hidefumi KAWASAKI
An explicit formula of the Shapley value for a cooperative game induced from the conjugate point

MHF2005-34 Hideki MURAKAWA
A regularization of a reaction-diffusion system approximation to the two-phase Stefan problem

MHF2006-1 Masahisa TABATA
Numerical simulation of Rayleigh-Taylor problems by an energy-stable finite element scheme

MHF2006-2 Ken-ichi MARUNO & G R W QUISPEL
Construction of integrals of higher-order mappings

MHF2006-3 Setsuo TANIGUCHI
On the Jacobi field approach to stochastic oscillatory integrals with quadratic phase function

MHF2006-4 Kouji HASHIMOTO, Kaori NAGATOU & Mitsuhiro T. NAKAO
A computational approach to constructive a priori error estimate for finite element approximations of bi-harmonic problems in nonconvex polygonal domains

MHF2006-5 Hidefumi KAWASAKI
A duality theory based on triangular cylinders separating three convex sets in \( \mathbb{R}^n \)

MHF2006-6 Raimundas VIDŪNAS
Uniform convergence of hypergeometric series
MHF2006-7  Yuji KODAMA & Ken-ichi MARUNO
N-Soliton solutions to the DKP equation and Weyl group actions

MHF2006-8  Toru KOMATSU
Potentially generic polynomial

MHF2006-9  Toru KOMATSU
Generic sextic polynomial related to the subfield problem of a cubic polynomial

MHF2006-10 Shu TEZUKA & Anargyros PAPAGEORGIOU
Exact cubature for a class of functions of maximum effective dimension

MHF2006-11 Shu TEZUKA
On high-discrepancy sequences

MHF2006-12 Raimundas VIDŪNAS
Detecting persistent regimes in the North Atlantic Oscillation time series

MHF2006-13 Toru KOMATSU
Tamely Eisenstein field with prime power discriminant

MHF2006-14 Nalini JOSHI, Kenji KAJIWARA & Marta MAZZOCCO
Generating function associated with the Hankel determinant formula for the solutions of the Painlevé IV equation