Type II blow-up mechanisms in a semilinear heat equation with Lepin exponent

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Abstract

We are concerned with blow-up mechanisms in a semilinear heat equation:

$$u_t = \Delta u + |u|^{p-1}u, \qquad x \in \mathbf{R}^N,\ t > 0,$$

where $p > 1$ is a constant. In the present article we describe formal asymptotic expansions of some peculiar type II blow-up solutions for $p = p_{\mathrm{L}} := 1 + 6/(N-10)$, $N \geq 11$. They are much different from existing results and their blow-up mechanisms are strongly influenced on space dimensions. The equations that dominate the blow-up mechanisms are obtained for high dimensions $N \geq 23$, low dimensions $N \leq 21$, and the threshold dimension $N = 22$, respectively.

Key words: type II blow-up; matched asymptotic expansions; threshold dimension
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1 Introduction and main results

1.1 Overview on blow-up rate

In the present article we discuss blow-up behavior for a semilinear heat equation:

$$u_t = \Delta u + |u|^{p-1}u, \quad x \in \mathbf{R}^N,\ t > 0, \tag{1.1a}$$

$$u(x,0) = u_0, \quad x \in \mathbf{R}^N, \tag{1.1b}$$

where Δ denotes the Laplacian in $\mathbf{R}^N$, $p > 1$ is a constant, and u_0 is a bounded function in $\mathbf{R}^N$. Local-in-time existence of a unique classical solution of (1.1) is well known. As usual, we say that a solution u of (1.1) blows up in a finite time T if

$$\limsup_{t\to T} \|u(\cdot,t)\|_{L^\infty(\mathbf{R}^N)} = +\infty. \tag{1.2}$$

Various sufficient conditions on given data for blow-up in finite time are known. For example, if u_0 is nonnegative and satisfies

$$\liminf_{|x|\to\infty} |x|^{2/(p-1)} u_0(x) > \mu^{-1/(p-1)} \tag{1.3}$$

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then the solution of (1.1) blows up in finite time, where $-\mu < 0$ is the principal eigenvalue of the Dirichlet Laplacian in a unit ball. See [19, Chapter II] and references therein.

In the present article, we study the blow-up rate of $\|u(\cdot,t)\|_{L^\infty(\mathbf{R}^N)}$ as t approaches the blow-up time T. It is well known that the self-similar rate is the slowest rate. Namely, for any blow-up solution u, there is a positive constant C such that

$$\|u(\cdot,t)\|_{L^\infty(\mathbf{R}^N)} \geq C(T-t)^{-1/(p-1)}, \quad 0<t<T.$$

It is far from obvious whether or not the corresponding upper estimate holds. A blow-up is called of **type I** if there exists a positive constant K such that

$$\|u(\cdot,t)\|_{L^\infty(\mathbf{R}^N)} \leq K(T-t)^{-1/(p-1)}, \quad 0<t<T; \tag{1.4}$$

whereas the blow-up is called of **type II** otherwise, i.e.,

$$\limsup_{t\nearrow T}(T-t)^{1/(p-1)}\|u(\cdot,t)\|_{L^\infty(\mathbf{R}^N)} = +\infty. \tag{1.5}$$

Unlike type I blow-up, exact rates of type II blow-up may not be unique, since (1.5) states only that the rate is faster than that of type I blow-up. In other words, singularity mechanisms of type II blow-up do not obey the self-similar structure of equation (1.1a) and several kinds of non-self-similar mechanisms can exist. For this reason it is not easy to determine the exact rates of type II blow-ups.

Classification of blow-up solutions in terms of blow-up rate has been considered as a first step toward understanding singularity formations for equation (1.1a), but it is still not yet complete even for radially symmetric solutions. This fact reflects rich structures of equation (1.1a). It is well known that the Joseph–Lundgren exponent:

$$p_{\mathrm{JL}} := \begin{cases} +\infty, & N \leq 10, \\ 1+\dfrac{4}{N-4-2\sqrt{N-1}}, & N \geq 11 \end{cases} \tag{1.6}$$

plays an important role. Throughout the present article, we use the following notations:

$$\beta = \frac{1}{p-1}; \tag{1.7a}$$

$$\gamma = \frac{N-2-\sqrt{16\beta^2-8(N-4)\beta+(N-2)(N-10)}}{2}. \tag{1.7b}$$

Notice that $\gamma>0$ is the smaller root of the quadratic equation:

$$\gamma^2-(N-2)\gamma+2(N-2\beta-2)(\beta+1)=0 \tag{1.8}$$

if $N\geq 11$ and $p>p_{\mathrm{JL}}$.

Herrero and Velázquez [9, 10] proved that, as long as $N \geq 11$ and $p_{\mathrm{JL}} < p$, type II blow-up actually occurs. They constructed type II blow-up solutions $\{u_{\ell,\mathrm{HV}}\}_{\ell\in\Lambda}$, $\Lambda\subset\mathbf{N}$, (which we call *HV solutions*) satisfying $\|u_{\ell,\mathrm{HV}}(\cdot,t)\|_\infty = u_{\ell,\mathrm{HV}}(0,t)$ and

$$C_1\,(T-t)^{-\beta-2\beta\omega_\ell} \leq u_{\ell,\mathrm{HV}}(0,t) \leq C_2\,(T-t)^{-\beta-2\beta\omega_\ell} \tag{1.9a}$$

$$\text{with} \quad \omega_\ell = \frac{\lambda_\ell}{\gamma-2\beta} > 0, \quad \lambda_\ell = \ell - \frac{\gamma}{2} + \beta \tag{1.9b}$$

for some constants $C_1, C_2 > 0$. Based upon the construction of HV solutions, further progress has been established in Matano [13] and Mizoguchi [18], independently. They proved, except for certain numbers of p, that if a radial solution blows up in finite time and the blow-up is of type II, then its actual blow-up rate coincides with that of $u_{\ell,\mathrm{HV}}$ for some $\ell \in \Lambda$. In this sense HV solutions describe blow-up mechanisms that are common among many type II blow-up solutions.

Whether or not there exists a type II blow-up solution for $p = p_{\mathrm{JL}}$ has been long remained open. The author has recently shown in [20] that type II blow-up may occur, at least in the sense of formal asymptotic expansions, for the case where $p = p_{\mathrm{JL}}$. This last result is the lowest possible with respect to the condition on p as far as nonnegative radially symmetric solutions are concerned. Indeed, Matano and Merle [14,15] and Mizoguchi [17] proved that if $p_{\mathrm{S}} < p < p_{\mathrm{JL}}$, type II blow-up cannot occur for radial solutions under mild assumptions on initial data, where p_{S} stands for the Sobolev exponent:

$$p_{\mathrm{S}} := \begin{cases} +\infty, & N = 1, 2, \\ \dfrac{N+2}{N-2}, & N \geq 3. \end{cases} \tag{1.10}$$

We note that, as long as $1 < p < p_{\mathrm{S}}$, the blow-up for every solution of (1.1a) is of type I even for non-radial or sign-changing solutions [7]. The Sobolev critical case, $p = p_{\mathrm{S}}$, is highly exceptional: There exists sign-changing type II blow-up solutions [2, 21] for $N = 3, 4, 5, 6$, but does not for nonnegative case [14, 15, 17]. It should also be noticed that, Matano and Merle [16] proved the existence of type II blow-up solutions of (1.1a) for p greater than a certain number $p^{**} (\geq p_L$ cf. (1.13) below) by a totally different method from the direct construction due to [9, 10].

1.2 A problem concerning neutral eigenvalues

Now, we shall turn our attention to the case $p > p_{\mathrm{JL}}$. Roughly speaking, the results of [13, 18] show that any radial type II blow-up solutions behave like one of HV solutions as the blow-up time is approached if p doesn't coincide with some exceptional values: The condition on p requires

$$n - \frac{\gamma}{2} + \frac{1}{p-1} \neq 0 \qquad \text{for all } n = 0, 1, ... \tag{1.11}$$

This last assumption on p means that the linearized operator about the singular stationary solution

$$\Phi_\infty(r) = c_* r^{-2\beta} \ \text{ with } \ c_* = \{2\beta\,(N - 2 - 2\beta)\}^\beta \tag{1.12}$$

in the self-similar variables (cf. (2.1) below) has no neutral eigenvalue. The exceptional values are distributed on the interval $[p_{\mathrm{JL}}, p_{\mathrm{L}}]$. Here and hereafter, we shall denote by p_{L} the Lepin exponent;

$$p_{\mathrm{L}} := \begin{cases} +\infty, & N \leq 10, \\ 1 + \dfrac{6}{N-10}, & N \geq 11. \end{cases} \tag{1.13}$$

A question that we address in the present article is to ask whether or not there exist type II blow-up solutions that behave at the singularities in a completely different manner from the HV solutions. The question can be rephrased as follows.

Does there exist a type II blow-up solution with exact blow-up rate different from (1.9) *if the extra assumption* (1.11) *on* p *is removed?*

The simplest value of p for which the assumption (1.11) fails is $p = p_{\mathrm{L}}$, $N \geq 11$. In this case there holds

$$2 - \frac{\gamma}{2} + \frac{1}{p-1} = 0.$$

We will construct such type II blow-up solutions in the sense of formal asymptotic expansions, consequently giving an affirmative answer to this question. Particularly interesting fact is that there are three types of blow-up mechanisms in accordance with spacial dimension; $11 \leq N \leq 21$, $N = 22$, and $N \geq 23$. Every blow-up mechanisms of the type II blow-up solutions are different from that of the HV solutions.

In the present article, we employ standard notations in asymptotic analysis; $\sim, \ll, \gg$, i.e., $f(\tau) \ll g(\tau)$ if $f(\tau) = o(g(\tau))$ and $f(\tau) \sim g(\tau)$ if $f(\tau) = g(\tau)(1 + o(1))$ as $\tau \to \infty$.

1.3 Main results

Construction of type II blow-up solutions for $p = p_{\mathbf{L}}$. (A) *Assume* $11 \leq N \leq 21$ *and* $p = p_L$ *hold. Then there exists a radial blow-up solution* u *of* (1.1a) *satisfying:*

$$\|u(\cdot, t)\|_\infty \sim C(T-t)^{-1/(p-1)} |\log(T-t)| \tag{1.14}$$

as $t \nearrow T$*, where* $C > 0$ *is a constant depending only on* N*.*
(B) *Assume* $N = 22$ *and* $p = p_L$ *hold. Then there exists a radial blow-up solution* u *of* (1.1a) *satisfying:*

$$\|u(\cdot, t)\|_\infty \sim C(T-t)^{-1/(p-1)} |\log(T-t)| \log|\log(T-t)| \tag{1.15}$$

as $t \nearrow T$*, where* $C > 0$ *is a constant.*
(C) *Assume* $N \geq 23$ *and* $p = p_L$ *hold. Then there exists a radial blow-up solution* u *of* (1.1a) *satisfying:*

$$\|u(\cdot, t)\|_\infty \sim C(T-t)^{-1/(p-1)} |\log(T-t)|^{(N-10)/12} \tag{1.16}$$

as $t \nearrow T$*, where* $C > 0$ *is a constant depending only on* N*.*

1.3.1 A few remarks on the main results

Remark 1.1. (Exact rates of type II blow-ups). Notice that the blow-up rates as in (1.14)-(1.16) are faster than the type I rate only by logarithmic factor, which are clearly different from (indeed slower than) the blow-up rates of HV solutions.

Remark 1.2. (The existence of the threshold dimension). The logarithmic factors in (1.14)-(1.16) come from corrective terms arising in the linearization about the singular stationary solution $\Phi_\infty(|y|)$ in the self-similar variables. The dominant factor of the nonlinear term is quite delicate on space dimension N. The asymptotics of solutions at the blow-up time differ accordingly. Such a result is known, in the sense of formal asymptotic expansions, in a different context [1], where the formation of type II singularity for a k-corotational harmonic map heat flow from $\mathbf{R}^d$ to a unit sphere is discussed. Among other results, the author of [1] described blow-up up mechanisms driven by neutral eigenvalue and different according to high and low dimensions, but never corresponding to the threshold dimension since it is not a physical dimension for any k, $d \in \mathbf{N}$.

The semilinear heat equation (1.1a) has a real parameter p as well as the space dimension $N \in \mathbf{N}$. That's why there is a situation where a neutral eigenvalue exists and the threshold dimension becomes a positive integer at the same time. In our problem, the result corresponding to the threshold dimension $N = 22$ is described in (B) above.

In general, it is believed that the exponential nonlinear heat equation $u_t = \Delta u + e^u$ could contain many analogy to (1.1a). However, this is not the case, since the linearized operator about the singular stationary solution $\tilde{\Phi}_\infty(r) := \log(2(N-2)/r^2)$ (in the self-similar variables) has a neutral eigenvalue if and only if $N = 10$. Thus threshold phenomenon like (1.14)-(1.16) cannot occur.

1.3.2 Organization of the article

In the next section we roughly describe a way leading to the main results. Some comparison with related results are also discussed there. In §3 we demonstrate in detail the way of construction in the sense of formal asymptotic expansions. It is partly inspired by [1]. As for the technical aspect, however, the author believes that the fundamental framework of the equations dominating the dynamics is presented here much more delicately than in [1]. The formal construction is a first step toward a full proof of the existence of the sough-for solutions. The argument in §3 is also expected to play an important role in proving rigorously the existence of blow-up solutions satisfying (A)–(C). Proofs of some technical results are gathered in §4.

2 Description by self-similar variables

2.1 Some preliminary results

Restricting ourselves in the radial case, we focus on the local behavior of solutions around $x = 0$. The description of our main results is better understood by using the backward self-similar variables:

$$\Phi(y,\tau) = (T-t)^\beta u(x,t), \tag{2.1a}$$

$$y = \frac{x}{\sqrt{T-t}}, \quad \tau = -\log(T-t). \tag{2.1b}$$

The new unknown function Φ then satisfies the rescaled equation:

$$\Phi_\tau = \Delta_y \Phi - \frac{y \cdot \nabla_y \Phi}{2} - \beta\Phi + |\Phi|^{p-1}\Phi \quad \text{in } \mathbf{R}^N \times (-\log T, \infty), \tag{2.2}$$

where $\nabla_y = {}^t(\partial_{y_1}, ..., \partial_{y_N})$ and $\Delta_y = \sum_{k=1}^N \partial_{y_k}^2$. Notice that $\Phi_\infty(r)$ as in (1.12) with $r = |y|$ is an unbounded stationary solution of (2.2). Classification of type I and type II blow-ups is now encoded to asking boundedness of function Φ as $\tau \to \infty$. Since u is nonnegative and radial in x, we may rewrite (2.2) as

$$\Phi_\tau = \Phi_{rr} + \left(\frac{N-1}{r} - \frac{r}{2}\right)\Phi_r - \beta\Phi + \Phi^p, \tag{2.3}$$

where $r = |y|$. Here, we have used abuse of notation: $\Phi(r,\tau) = \Phi(y,\tau)$ just for simplicity.

Equation (2.2) describes the local profile, i.e., the blow-up asymptotics of $u(x,t)$ in the backward parabola below the space-time point $(x,t) = (0,T)$. As long as $p > p_S$, type II blow-up for radial solutions of (1.1a) is characterized in terms of the convergence to the singular stationary solutions [15]. Among other results, the results of [15] are stated in our notations as follows. Though similar results were proved also for bounded balls, we mention those results only for the whole space $\mathbf{R}^N$ for simplicity.

Proposition 2.1. *(Matano and Merle* [15, Theorems 3.1 and 3.2]*). Let $p_S < p < \infty$. If a radial solution u of* (1.1a) *blows up at $t = T < \infty$, then the local blow-up profile:*

$$\Phi^*(y) := \lim_{\tau\to\infty} \Phi(y,\tau)$$

exists locally uniformly in $y \in \boldsymbol{R}^N \setminus \{0\}$ and is either a bounded stationary solution of (2.3) *or the singular stationary solutions $\pm\Phi_\infty(|y|)$. Moreover, the following conditions are equivalent:*

(a) *the blow-up is of type II;* (b) $\lim_{t\to T}(T-t)^{1/(p-1)}\|u(\cdot,t)\|_\infty = \infty$;

(c) $\Phi^*(y) = \Phi_\infty(|y|)$ *or* $-\Phi_\infty(|y|)$; (d) $\displaystyle\lim_{x\to 0}\frac{u(x,T)}{\Phi_\infty(|x|)} = 1$ *or* -1,

where the term $u(x,T) := \lim_{t\to T} u(x,t)$ denotes the global blow-up profile.

Proposition 2.1 implies, in particular, that the Dirac delta profile associated with type II blow-up cannot occur for $p > p_{\mathrm{S}}$. It is quite notable difference from the case $p = p_{\mathrm{S}}$ [2,21]. More precisely, it was proven in [15, Corollary 4.2] that if $p > p_{\mathrm{S}}$ and if there exists a constant $M > 0$ such that

$$\limsup_{t\to T} |u(x,t)| \le M \qquad \text{for every } x \ne 0,$$

then u is bounded on $\mathbf{R}^N \times [0,T)$.

Due to the Proposition 2.1, it is sufficient to show that there exists a solution $\Phi(y,\tau)$ of (2.3) that converges to $\Phi_\infty(|y|)$. Let us set

$$v(r,\tau) = \Phi(r,\tau) - \Phi_\infty(r). \tag{2.4}$$

It is readily seen that ψ solves the equation

$$v_\tau = \frac{1}{r^{N-1}\rho}\frac{\partial}{\partial r}\left(r^{N-1}\rho\frac{\partial v}{\partial r}\right) - \beta v + \frac{pc_*^{p-1}}{r^2}v + f(v) \equiv -\mathcal{A}v + f(v) \tag{2.5a}$$

$$f(\phi) := (\phi + \Phi_\infty)^p - \Phi_\infty^p - \frac{pc_*^{p-1}}{r^2}\phi. \tag{2.5b}$$

The linearized operator A, that is initially defined in the set of smooth functions with compact support, may be extended to a unique lower-bounded self-adjoint operator if $N \ge 11$ and $p \ge p_{\mathrm{JL}}$. The result is stated as follows (cf. [9, Lemma 2.3] and [20]).

Proposition 2.2. *Assume that $N \geq 11$ and $p \geq p_{JL}$ hold. Then the spectrum of A consists only of simple eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$,*

$$\lambda_n = n - \frac{\gamma}{2} + \beta, \qquad n = 0, 1, 2, \dots \tag{2.6}$$

Eigenfunctions of A associated with eigenvalues λ_n are given by

$$\phi_n(r) = c_n r^{-\gamma} M\left(-n, -\gamma + \frac{N}{2}; \frac{r^2}{4}\right), \qquad n = 0, 1, 2, \dots, \tag{2.7a}$$

where

$$M(a, b; z) = 1 + \sum_{j=1}^{\infty} \frac{(a)_j}{j!(b)_j} z^j \quad \text{with} \quad (a)_m = \prod_{j=0}^{m-1} (a + j) \tag{2.7b}$$

is a Kummer function and where $c_n > 0$, $n = 0, 1, \dots$, are constants such that $\|\phi_n\| = 1$. Moreover, the eigenfunctions behave asymptotically as

$$\phi_n(r) = c_n r^{-\gamma}(1 + o(1)) \quad \text{ as } r \to 0; \tag{2.8a}$$

$$\phi_n(r) = \tilde{c}_n r^{-\gamma+2n}(1 + o(1)) \quad \text{ as } r \to \infty, \tag{2.8b}$$

where $\tilde{c}_n$ are constants such that $(-1)^n \tilde{c}_n > 0$ for $n = 0, 1, 2, \dots$ Furthermore, the constants c_n and $\tilde{c}_n$ in (2.8) are represented as

$$c_n = \left(\frac{\Gamma(-\gamma + N/2 + n)}{\Gamma(-\gamma + N/2)^2 \Gamma(n + 1)}\right)^{1/2} 2^{\gamma - N/2 + 1/2}, \tag{2.9}$$

$$\tilde{c}_n = \frac{(-1)^n}{(-\gamma + N/2)_n} \left(\frac{\Gamma(-\gamma + N/2 + n)}{\Gamma(-\gamma + N/2)^2 \Gamma(n + 1)}\right)^{1/2} 2^{\gamma - N/2 + 1/2}, \tag{2.10}$$

respectively, where Γ stands for the standard Gamma function.

Remark 2.3. (i) A simple computation shows that $\lambda_0 < \lambda_1 < 0$. Moreover, $\lambda_2 > 0$ (resp. $\lambda_2 = 0$) if and only if $p > p_{\mathrm{L}}$ (resp. $p = p_{\mathrm{L}}$). As a matter of fact, higher eigenvalues vanish by choosing suitable values of p's between p_{JL} and p_{L}. When $p = p_{\mathrm{L}}$, we have $\lambda_n = n - 2$ for all n. (ii) By classical results on orthogonal polynomials, the eigenfunctions are in fact written by associated Laguerre polynomials $L_n^{(\nu)}(z)$ with degree ν:

$$\phi_n(r) = c_n r^{-\gamma} \frac{\Gamma(\alpha + 1)\, n!}{\Gamma(\alpha + n + 1)} L_n^{(\nu)}\left(\frac{r^2}{4}\right) \quad \text{with} \quad \nu = -\gamma + \frac{N}{2} - 1. \tag{2.11}$$

2.2 A brief description of the main results

As the eigenfunctions $\{\phi_n\}_{n=0}^{\infty}$ form a complete orthonormal system of $L^2_{r,\rho}(\mathbf{R}^N)$, we may expand a solution v of (2.5) to the Fourier series:

$$v(\cdot, \tau) = [a_0(\tau)\phi_0 + a_1(\tau)\phi_1] + a_2(\tau)\phi_2 + \sum_{j=3}^{\infty} a_j(\tau)\phi_j, \tag{2.12}$$

where the convergence takes place in the norm of $L^2_{r,\rho}(\mathbf{R}^N)$. Notice that $a_0(\tau), a_1(\tau)$ correspond to the "unstable modes", $a_2(\tau)$ to the "neutral one", and $a_3(\tau), a_4(\tau), \ldots$ to the "stable ones". In any space dimension, our results imply that the neutral mode eventually dominates in (2.12) and consequently

$$\Phi(y,\tau) \sim \Phi_\infty(|y|) + a_2(\tau)\phi_2(y) \quad \text{ as } \tau \to \infty \tag{2.13}$$

in the set $\{\varepsilon(\tau)^\theta \leq |y|\}$ with any $0 < \theta < 1$.

This situation may remind some readers of the one in the Sobolev subcritical case. We just recall the well-known results by [3–5, 8], focusing the one-dimensional case there for simplicity. Blow-up of a solution of (1.1a) is always of type I [5]. It follows that the rescaled solution $\Phi(y,\tau)$ converges to a unique bounded stationary solution $\kappa := \beta^\beta$ of (2.2) locally uniformly in $\mathbf{R}$ [4]. Further results on the convergence were obtained by [3] and [8] simultaneously and independently. Let us set $\psi := \Phi(y,\tau) - \kappa$, which solves

$$\psi_\tau - \Delta\psi + \frac{y \cdot \nabla\psi}{2} - \psi = \frac{p}{2\kappa}\psi^2 + O\left(\psi^3\right) \tag{2.14}$$

By a classical result, the linearized operator $\mathcal{L}\psi = -\Delta\psi + \frac{y\cdot\nabla\psi}{2} - \psi$ may be defined as a self-adjoint operator in $L^2_\rho(\mathbf{R}) := L^2(\mathbf{R}, \rho dy)$, where $\rho = \exp(-y^2/4)$, and its spectrum consists only of eigenvalues $\{\mu_n\}_{n=0}^\infty \subset \mathbf{R}$,

$$\mu_0 < \mu_1 < \mu_2 = 0 < \mu_3 < \cdots$$

Eigenfunctions $\{h_n\}_{n=0}^\infty$, which consist of Hermite polynomials, form a complete orthonormal system of $L^2_\rho(\mathbf{R})$. Following [3] (with slightly different notations), we shall write

$$\psi(\cdot,\tau) = [b_0(\tau)h_0 + b_1(\tau)h_1] + b_2(\tau)h_2 + [b_3(\tau)h_3 + b_4(\tau)h_4 + \cdots]$$

Notice that $b_0(\tau), b_1(\tau)$ correspond to the "unstable modes", $b_2(\tau)$ to the "neutral one", and $b_3(\tau), b_4(\tau), \ldots$ to the "stable ones". If the standard center manifold theory were applied, we expect that the unstable and stable modes would be negligible and

$$\Phi(y,\tau) \sim \kappa + b_2(\tau)h_2(y) \tag{2.15}$$

as $\tau \to \infty$. The situation thus apparently look like (2.13). Guided by the center manifold theory and using particular structures of equation (1.1a), the authors of [3] proved that (2.15) actually holds unless the convergence is exponentially fast.

There are, however, two major differences: First, we know that *any* solution $\Phi(y,\tau)$ of (2.2) converges to κ locally uniformly in $\mathbf{R}$. On the other hand, we are trying to *construct* a solution Φ of (2.2) which converges to Φ_∞ without any information about blow-up rate. As is mentioned before, there exist type II blow-up solutions due to [9, 10] for any $p > p_{\mathrm{JL}}$, which covers $p = p_{\mathrm{L}}$. However, any of them does not correspond to the convergence along with a neutral mode (i.e., as in (2.13)) but with a stable mode.

The second notable difference consists in the scale of space-time regions in which the convergence takes place. Since type I blow-up is of self-similar nature, the convergence in the self-similar variables takes place in mono-scale. On the other hand, the occurrence of type II blow-up is associated with the presence of multi-scales. Namely, the scale of outer part is given by the self-similar variables (y,τ), whereas that of inner part is given by (ξ,τ) with $\xi = y/\varepsilon(\tau)$. Here, $\varepsilon(\tau)$ denotes the size of the boundary layer caused by the rescaling $y \mapsto \xi$. Type II blow-up is deeply related to the existence of such a function $\varepsilon(\tau)$. There can be a possible variety of $\varepsilon(\tau)$ in accordance with the way of convergence of $\Phi(y,\tau)$ to Φ_∞ in the outer part. What we show in the present article is that there exists $\varepsilon(\tau)$ associated with the neutral eigenvalue. Unlike the case of stable mode, the convergence rate along with the neutral eigenvalue do depend on space dimensions. This is due to the fact that the convergence to Φ_∞ along with the neutral eigenvalue is not governed by the standard linearized equation but dominated by a linear equation with suitable forcing term, whose leading term is determined by the balance between the inner and outer scales. The precise outer expansions are obtained as follows:

$$\begin{aligned} (11 \le N \le 21): \quad & \Phi(y,\tau) \sim \Phi_\infty(|y|) - C_1\tau^{-12/(N-10)}\phi_2(|y|) \quad \text{for } \varepsilon(\tau) \ll |y|; \\ & \varepsilon(\tau) \sim B_1\tau^{-3/(N-10)} \end{aligned} \tag{2.16a}$$

$$\begin{aligned} (N = 22): \quad & \Phi(y,\tau) \sim \Phi_\infty(|y|) - \frac{C_2}{\tau\log\tau}\phi_2(|y|) \quad \text{for } \varepsilon(\tau)|\log\varepsilon(\tau)|^{1/4} \ll |y|; \\ & \varepsilon(\tau) \sim B_2\,(\tau\log\tau)^{-1/4} \end{aligned} \tag{2.16b}$$

$$\begin{aligned} (23 \le N): \quad & \Phi(y,\tau) \sim \Phi_\infty(|y|) - \frac{C_3}{\tau}\phi_2(|y|) \quad \text{for } \varepsilon(\tau) \ll |y|; \\ & \varepsilon(\tau) \sim B_3\tau^{-1/4}, \end{aligned} \tag{2.16c}$$

where B_i and C_i, $i = 1,2,3$, are positive constants. Notice that the outer region for $N = 22$ is smaller than the other ones. On the other hand, the dynamics in the inner region $\{|y| \ll 1\}$ is described, in any dimension, as

$$\Phi(y,\tau) \sim \varepsilon(\tau)^{-2\beta}U(|\xi|) \tag{2.17a}$$

$$\sim \Phi_\infty(|y|) - h\varepsilon(\tau)^{-2\beta}\left(\frac{|y|}{\varepsilon(\tau)}\right)^{-\gamma} \quad \text{for } \varepsilon(\tau) \ll |y| \ll 1 \tag{2.17b}$$

as $\tau \to \infty$, where U is a bounded function of one variable and $h > 0$ is a constant. The main result is then obtained by substituting the asymptotic formulas (2.16) to (2.17a) with the original variables.

3 Blow-up mechanisms driven by neutral eigenvalues

In this section we derive the main results by means of the matched asymptotic expansions. Suppose that an inner layer near the origin appears in our sought-for solution $\Phi(r,\tau)$ of (2.2), where sharp changes in Φ arise when $\tau \to \infty$. Let $\varepsilon(\tau)$ denote the size of the inner layer, which is a priori unknown. We assume

$$\varepsilon(\tau),\ \dot{\varepsilon}(\tau) \to 0 \quad \text{as } \tau \to \infty. \tag{3.1}$$

Moreover, we assume

$$\dot{\varepsilon}(\tau) \ll \varepsilon(\tau) \quad \text{as } \tau \to \infty. \tag{3.2}$$

To see the dynamics near the origin, we introduce inner variables $(U(\xi,\tau),\xi)$ as follows:

$$\Phi(y,\tau) = \varepsilon(\tau)^{-2\beta}U(\xi,\tau), \quad \xi = \frac{y}{\varepsilon(\tau)}. \tag{3.3}$$

A direct computation then shows that

$$\varepsilon(\tau)^2 U_\tau = \Delta_\xi U + U^p + \left(2\varepsilon(\tau)\dot{\varepsilon}(\tau) - \varepsilon(\tau)^2\right)\left(\frac{\xi\cdot\nabla_\xi U}{2} + \beta U\right). \tag{3.4}$$

In view of (3.1), we naturally infer that the leading term of U as $\tau \to \infty$ would be given by a bounded stationary solution of (1.1a) as long as $\xi = o(1/\varepsilon(\tau))$.

Proposition 3.1. *([12, Lemma 4.3]) For any $\alpha > 0$, there exists a unique solution U_α of*

$$\frac{d^2U}{dr^2} + \frac{N-1}{r}\frac{dU}{dr} + U^p = 0 \quad \text{for } r>0, \qquad U(0)=\alpha, \quad U'(0)=0. \tag{3.5}$$

Moreover, if $p > p_{JL}$ with $N \geq 11$, there holds

$$U_\alpha(r) = U_\infty(r) - h_\alpha r^{-\gamma} + o(r^{-\gamma}) \qquad \text{as } n \to \infty, \tag{3.6}$$

where $h_\alpha = h_1\alpha^{1-\gamma(p-1)/2}$ with h_1 being a positive constant.

Due to (3.3) and (3.4), it is natural to construct an inner solution of the form:

$$\Phi_{\mathrm{inn}}(r,\tau) := \varepsilon(\tau)^{-2\beta}U_\alpha\left(\frac{r}{\varepsilon(\tau)}\right)$$

where $\alpha > 0$ is a free constant to be selected later on. The asymptotic behavior (3.6) of U_α then implies

$$\Phi_{\mathrm{inn}}(r,\tau) \sim \Phi_\infty(r) - h_\alpha\varepsilon(\tau)^{\gamma-2\beta}r^{-\gamma} \quad \text{for } \varepsilon(\tau) \ll r \ll 1. \tag{3.7}$$

We will construct an outer solution $\Phi_{\mathrm{out}}(y,\tau)$ of (2.3) satisfying (3.7). From now on, we concentrate on the case where $p = p_{\mathrm{L}}$, so that

$$\lambda_2 = 0 \tag{3.8}$$

(cf. (2.6)), or equivalently $\gamma = 2\beta + 4$. We consider the function v (cf. (2.4)) and its Fourier expansions (2.12). Required from the inner expansions (3.7), we will look for an outer solution Φ_{out} satisfying

$$\Phi_{\mathrm{out}}(r,\tau) \sim \Phi_\infty(r) - B\varepsilon(\tau)^{\gamma-2\beta}r^{-\gamma} \quad \text{as } r \to 0 \tag{3.9}$$

for some constant $B > 0$, where $\gamma - 2\beta = 4$ due to (2.6) and (3.8).

Throughout of the present article, we assume:

Assumption 3.1. *The evolution of function v as $\tau \to \infty$ is governed by the eigenfunction associated with the neutral eigenvalue in (2.12), i.e.,*

$$|a_n(\tau)| \ll |a_2(\tau)| \ll 1, \quad \forall n \neq 2. \tag{3.10}$$

Under the Assumption 3.1, the condition (3.9) implies that

$$a_2(\tau) = D\varepsilon(\tau)^4(1+o(1)) \quad \text{as } \tau \to \infty, \tag{3.11}$$

where $D \neq 0$ is a constant. The Fourier coefficients in (2.12) solve the ODE:

$$\dot{a}_n(\tau) = -\lambda_n a_n(\tau) + \langle f(v(\cdot,\tau)), \phi_n\rangle \tag{3.12}$$

for $n = 0, 1, 2, \ldots$ In particular,

$$\dot{a}_2(\tau) = \langle f(v(\cdot,\tau)), \phi_2\rangle. \tag{3.13}$$

Given a continuous function $\phi(r)$ on $\mathbf{R}_+$ such that $\phi(r) = O\left(r^{-\gamma}\right)$ as $r \to 0$, we set

$$\langle f\left(v(\cdot,\tau)\right), \phi\rangle = \left(\int_0^L + \int_L^\infty\right) f(v(r,\tau))\phi(r)r^{N-1}\rho(r)dr =: I_1 + I_2, \tag{3.14}$$

where $L = L(\tau)$ is a continuous function satisfying $\varepsilon(\tau) \ll L \ll 1$ as $\tau \to \infty$. We shall compute the asymptotics of the both integrals as $\tau \to \infty$. Using the inner expansion (3.7), we obtain

$$I_1 \sim \varepsilon(\tau)^{-2p\beta+N} \int_0^{L/\varepsilon(\tau)} \left[U_\alpha\left(\xi\right)^p - U_\infty\left(\xi\right)^p - \frac{pc_*^{p-1}}{\xi^2}\left(U_\alpha - U_\infty\right)(\xi)\right]\xi^{N-1}(\phi\rho)(\varepsilon(\tau)\xi)d\xi,$$

where the change of variable $r = \varepsilon(\tau)\xi$ has been used. Set $\hat{\phi}(0) = \lim_{r\to 0} r^\gamma\phi(r)$. Taking account that $(\phi\rho)\left(\varepsilon(\tau)\xi\right) \sim \hat{\phi}(0)\left(\varepsilon(\tau)\xi\right)^{-\gamma}(1+o(1))$ for $\varepsilon(\tau)\xi \leq L$, we obtain

$$I_1 \sim \hat{\phi}(0)\varepsilon(\tau)^{-2p\beta+N-\gamma} \int_0^{L/\varepsilon(\tau)} \left[U_\alpha\left(\xi\right)^p - U_\infty\left(\xi\right)^p - \frac{pc_*^{p-1}}{\xi^2}\left(U_\alpha - U_\infty\right)(\xi)\right]\xi^{N-1-\gamma}d\xi$$

as $\tau \to \infty$. It is essential to estimate the last integral. To proceed further, we divide our argument into three cases.

3.1 High dimension case: $N \geq 23$

By definition, we have $\lim_{\tau\to\infty} L/\varepsilon(\tau) = +\infty$. The asymptotic formula (3.6) of U_α yields

$$U_\alpha^p - (U_\infty)^p - \frac{pc_*^{p-1}}{\xi^2}\left(U_\alpha - U_\infty\right) \sim \frac{p(p-1)}{2!}c_*^{p-2}\xi^{-2\beta(p-2)}\left(U_\alpha - U_\infty\right)^2,$$

whence:

$$I_1 \sim \frac{1}{2}p(p-1)h_\alpha^2 c_*^{p-2}\hat{\phi}(0)\varepsilon(\tau)^{-2p\beta+N-\gamma}\int_0^{L/\varepsilon(\tau)} \xi^{-2\beta(p-2)-3\gamma+N-1}d\xi. \tag{3.15}$$

Since $\varepsilon(\tau) \ll L$ and $-2\beta(p-2) - 3\gamma + N - 1 = (N-22)/3 - 1 > -1$ for $N \geq 23$, the integral in (3.15) diverges as $\tau \to \infty$. L'Hôpital's rule then implies that

$$I_1 \sim C_2\hat{\phi}(0)\varepsilon(\tau)^{-2p\beta+N-\gamma}\left(\frac{L}{\varepsilon(\tau)}\right)^{-2\beta(p-2)-3\gamma N} = C_2\hat{\phi}(0)\varepsilon(\tau)^8 L^{(N-22)/3} \tag{3.16}$$

as $\tau \to \infty$, where $C_2 = 3p(p-1)h_\alpha^2 c_*^{p-2}/2(N-22) > 0$.

We next consider I_2. Due to (3.11), we have

$$f(v) \sim \frac{1}{2}p(p-1)c_*^{p-2}r^{-2\beta(p-2)}v^2.$$

We now claim that the leading term of the outer solution is determined by the neutral mode only, i.e.,

$$v(r,\tau) \sim a_2(\tau)\phi_2(r) \tag{3.17}$$

in the outer region $\varepsilon(\tau) \ll r$.

To verify (3.17), we show that the remainder term cannot yield the same order of magnitude. Let us write

$$v(r,\tau) = a_0(\tau)\phi_0 + a_1(\tau)\phi_1(r) + a_2(\tau)\phi_2(r) + Q(r,\tau), \tag{3.18a}$$

$$\langle Q(\cdot,\tau), \phi_k\rangle = 0 \quad (k = 0,1,2). \tag{3.18b}$$

The equation (2.5) for v then reads

$$Q_\tau = Q_{rr} + \left(\frac{N-1}{r} - \frac{r}{2}\right)Q_r - \beta Q + \frac{pc_*^{p-1}}{r^2}Q + f(v) - \sum_{k=0}^{2}\left(\dot{a}_k(\tau) - \lambda_k a_k(\tau)\right)\phi_k(r)$$

In view of Assumption 3.1 and (3.11), we expect that $Q(r,\tau) \sim \varepsilon(\tau)^\ell F(r)$ with $\ell \geq 4$ and

$$\begin{aligned}
&\sum_{k=0}^{2}\left(\dot{a}_k(\tau) - \lambda_k a_k(\tau)\right)\phi_k(r) + \ell\varepsilon(\tau)^{\ell-1}\dot{\varepsilon}(\tau)F \\
&\qquad = \varepsilon(\tau)^\ell\left[F'' + \left(\frac{N-1}{r} - \frac{r}{2}\right)F' - \beta F + \frac{pc_*^{p-1}}{r^2}F\right] + \\
&\qquad\qquad + \frac{1}{2}p(p-1)c_*^{p-2}r^{-2\beta(p-2)}\left\{\sum_{k=0}^{2}a_k(\tau)\phi_k(r) + \varepsilon(\tau)^\ell F\right\}^2 + \cdots.
\end{aligned}$$

Notice that $4 - 2\beta(p-2) = (N-1)/3 > 0$ for $p = p_{\mathrm{L}}$, whence the second term of the right-hand side is subdominant in the regions where $\varepsilon(\tau) \ll r \ll 1$ due to (3.11). We then obtain

$$F'' + \left(\frac{N-1}{r} - \frac{r}{2}\right)F' - \beta F + \frac{pc_*^{p-1}}{r^2}F = 0. \tag{3.19}$$

Due to classical ODE theory, every solution of (3.19) is expressed by a linear combination of two particular solutions. The one is given by $\phi_2(r)$. The other one grows exponentially as a function of $r^2/4$ as $r \to \infty$. We thus obtain $F(r) = C\phi_2(r)$ for some constant $C \in \mathbf{R}$, but then the orthogonality condition (3.18b) forces $C \equiv 0$. Consequently (3.17) follows.

We now substitute (3.17) to (3.14), to get

$$I_2 \sim \frac{p(p-1)}{2}c_*^{p-2}a_2(\tau)^2\int_L^\infty \phi_2(r)^2\phi(r)r^{-2\beta(p-2)+N-1}\rho dr.$$

Since $\phi_2(r)^2\phi(r) = O(r^{-3\gamma})$ as $r \to 0$ and $-2\beta(p-2)-3\gamma+N-1 = (N-22)/3-1 > -1$ for $N \geq 23$, the last integral converges as $L \to 0$. By (3.11) we then obtain

$$I_2 \sim C_{N,2}a_2(\tau)^2 \left(= C_{N,2}D^2\varepsilon(\tau)^8\right), \tag{3.20}$$

where

$$C_{N,2} = \frac{p(p-1)c_*^{p-2}}{2}\int_0^\infty \phi_2(r)^2\phi(r)r^{-2\beta(p-2)+N-1}e^{-r^2/4}dr. \tag{3.21}$$

Putting (3.16) and (3.20) together, we obtain

$$\begin{aligned}\langle f(v(\cdot,\tau)),\phi\rangle &\sim C_2\hat{\phi}(0)\varepsilon(\tau)^8L^{(N-22)/3} + C_{N,2}a_2(\tau)^2 \\ &\sim C_{N,2}a_2(\tau)^2\end{aligned}$$

as $\tau \to \infty$. Particularizing $\phi(r) = \phi_2(r)$ in (3.14), we obtain, from (3.13),

$$\dot{a_2}(\tau) = C^*_{N,2}a_2(\tau)^2(1+o(1)) \quad \text{ as } \tau \to \infty,$$

where $C^*_{N,2}$ is the positive constant obtained by substituting $\phi = \phi_2$ in (3.21). A tedious computation shows that $C^*_{N,2}$ is positive. It then follows that

$$a_2(\tau) = -\frac{1}{C^*_{N,2}\tau}(1+o(1)), \qquad \varepsilon(\tau) = \frac{1}{\left(C^*_{N,2}\tau\right)^{1/4}}(1+o(1)),$$

whence $\Phi(0,\tau) \sim \left(C^*_{N,2}\tau\right)^{(N-10)/12}$ as $\tau \to \infty$. Returning to the original variables, we conclude the desired result (1.16).

3.2 Low dimension case: $11 \leq N \leq 21$

In the low dimensional case, the integral in (3.15) is convergent, so:

$$I_1 \sim D_{N,2}\hat{\phi}(0)\varepsilon(\tau)^{-2p\beta+N-\gamma} = D_{N,2}\hat{\phi}(0)\varepsilon(\tau)^{\gamma}$$

as $\tau \to \infty$, where

$$D_{N,2} = C_1\int_0^\infty \left(U_\alpha(\xi)^p - U_\infty(\xi)^p - \frac{pc_*^{p-1}}{\xi^2}\left(U_\alpha - U_\infty\right)(\xi)\right)\xi^{N-1-\gamma}d\xi > 0$$

with $C_1 = 2^{-1}p(p-1)c_2h_\alpha^2c_*^{p-2}$.

On the other hand, Taylor approximation yields

$$\begin{aligned}I_2 &\sim \frac{p(p-1)c_*^{p-2}}{2}a_2(\tau)^2\int_L^\infty \phi_2(r)^2\phi(r)r^{-2\beta(p-2)+N-1}e^{-r^2/4}dr \\ &= O\left(\varepsilon(\tau)^8L^{-(22-N)/3}\right),\end{aligned}$$

where (3.11) has been used in the last step. Thus I_2 yields subdominant contribution compared to I_1. We have therefore obtained:

$$\langle f(v(\cdot,\tau)),\phi_2\rangle \sim D_{N,2}\varepsilon(\tau)^{\gamma} \quad \text{ as } \tau \to \infty. \tag{3.22}$$

As in the case of high dimensions, we want to show that the leading behavior of outer expansions is determined only by the neutral mode. To this end, we introduce a new variable:

$$W = r^\gamma v. \tag{3.23}$$

A direct computation shows that

$$W_\tau = W_{rr} + \left(\frac{N-1-2\gamma}{r} - \frac{r}{2}\right) W_r + 2W + g(W) \tag{3.24}$$

where $g(W) = r^\gamma f(r^{-\gamma} W)$.

Consider the Hilbert space $X := L^2\left(\mathbf{R}_+, r^{N-2\gamma-1}\rho dr\right)$. Notice that for $N = 13, 16, 19$, the number $N - 2\gamma$ is a positive integer, so the space X coincides with the space of all radial square integrable functions in $\mathbf{R}^{N-2\gamma}$ with respect to the weighted measure $\rho(|y|)dy$. The linear operator $\mathcal{L}$ defined by

$$-\mathcal{L}\psi = \psi'' + \left(\frac{N-1-2\gamma}{r} - \frac{r}{2}\right)\psi' + 2\psi, \quad \psi \in D(\mathcal{L}) = \{\varphi \in L^2_{\text{loc}}(0,\infty)\,; \varphi, \varphi' \in X\}.$$

is self-adjoint. Its spectrum consists of the same eigenvalues $\{\lambda_n\}_{n=0}^\infty$ as of A and eigenfunctions associated with λ_n are given by

$$\psi_n(r) = r^\gamma \phi_n(r), \tag{3.25a}$$

$$= c_n \frac{\Gamma(\nu+1)\, n!}{\Gamma(\nu+n+1)} L_n^{(\nu)}\left(\frac{r^2}{4}\right) \tag{3.25b}$$

(cf. (2.11)). They satisfy

$$\psi_n'' + \left(\frac{N-1-2\gamma}{r} - \frac{r}{2}\right)\psi_n' + (\lambda_n + 2)\psi_n = 0$$

and

$$\int_0^\infty \phi_n(r)\phi_m(r) r^{N-1}\rho dr = \int_0^\infty \psi_n(r)\psi_m(r) r^{N-1-2\gamma}\rho dr$$

for all $n, m = 0, 1, ...$ The Fourier expansion (2.12) reads

$$W(r,\tau) = a_0(\tau)\psi_0 + a_1(\tau)\psi_1(r) + a_2(\tau)\psi_2(r) + Q(r,\tau), \tag{3.26}$$

where $Q(r,\tau) = \sum_{k=3}^\infty a_k(\tau)\psi_k(r)$. What we want show is that:

$$W(r,\tau) \sim a_2(\tau)\psi_2(r) \quad \text{for } \varepsilon(\tau) \ll r \ll 1$$

as $\tau \to \infty$. To this end, we have to exclude the possibility that the remainder term $Q(r,\tau)$, as well as the first two terms in (3.26), could contribute to the leading term. This is not clear. Indeed, the technique to be presented below was developed in [20] in the case $p = p_{\text{JL}}$. In that paper, the remainder term corresponding to $Q(r,\tau)$ above does yield the same order of magnitude as of the leading term, e.g., for $r = \varepsilon(\tau)^a$ with $a \in (0,1)$.

In order to take the possible contribution into account, we shall consider the equation (3.24) in a weak sense. We first notice that (3.22) suggests replacing $g(W)$ by $D_{N,2}\varepsilon(\tau)^{\gamma}S(r)$ with the functional S defined by

$$\langle S, \psi\rangle = \psi(0) \quad \text{ for } \psi \in C\left([0,\infty)\right). \tag{3.27}$$

The functional S may be regarded as the Dirac mass over an Euclidean space of "fractional dimension". We now define a subspace V of X by

$$V := D(\mathcal{L}^2) = \{\psi \in X;\ \|\psi\|_V^2 := \|\psi\|^2 + \|\mathcal{L}^2\psi\|^2 < +\infty\}.$$

It is a Hilbert space endowed with canonical inner product. By the self-adjointness of $\mathcal{L}$, we have

$$(\psi, \psi_n)_V = (\psi, \psi_n)_X + \left(\mathcal{L}^2\psi, \mathcal{L}^2\psi_n\right)_X = \left(1+\lambda_n^4\right)(\psi, \psi_n)_X \tag{3.28}$$

for all n. Let us write

$$\Psi_n(r) = \frac{\psi_n(r)}{\sqrt{1+\lambda_n^4}}.$$

Since $\{\psi_n\}_{n=0}^{\infty}$ is a complete orthonormal system in X, so is $\{\Psi_n\}_{n=0}^{\infty}$ in V due to (3.28). The dual space V' plays a fundamental role in our asymptotic analysis. In the following, we define a map $\iota:\ X \to V'$ by

$$\langle \iota F, \psi\rangle = \int_0^{\infty} F(r)\psi(r) r^{N-2\gamma-1}\rho dr \quad \textit{for } \psi \in V. \tag{3.29}$$

Each element $F \in X$ may thus be regarded as an element of V' by identifying it with ιF, whence X is canonically embedded in V'.

Proposition 3.2. *A bounded linear functional T on V is expressed as*

$$T = \sum_{j=0}^{\infty} \langle T, \psi_j\rangle \iota\psi_j \quad \textit{in } V' \tag{3.30}$$

and there holds

$$\|T\|_{V'}^2 = \sum_{j=0}^{\infty} \frac{|\langle T, \psi_j\rangle|^2}{1+\lambda_j^4}. \tag{3.31}$$

Conversely, if the series in the right-hand side of (3.31) *converges for some linear functional $T = T_0$ defined in $C\left([0,\infty)\right)$, then T_0 may be uniquely extended to a bounded linear functional $\tilde{T}$ such that $\langle \tilde{T}, \psi_n\rangle = \langle T_0, \psi_n\rangle$ for each n.*

To keep the flow of the main argument, we shall postpone the proof of Proposition 3.2 to §4 and admit the statement here. We shall apply Proposition 3.2 for S as in (3.27). Since $\langle S, \psi_n\rangle = c_n$ by (2.8a) and (3.25b), the only thing that we have to check is that

$$\sum_{n=0}^{\infty} \frac{c_n^2}{1+\lambda_n^4} < +\infty, \tag{3.32}$$

where $\{c_n\} \subset \mathbf{R}$ is explicitly given in (2.9). Notice that $-\gamma + N/2 = (N-4)/6 =: 2 + \theta$ with $|\theta| \le 1$ for $N \le 22$ and $p = p_{\mathrm{L}}$. Recalling the well-known Stirling formula:

$$\Gamma(z) \sim \sqrt{2\pi} e^{-z} z^{z-1/2} \qquad \text{as } |z| \to \infty, -z \notin \mathbf{R}_+,$$

we have

$$\frac{\Gamma(n+2+\theta)}{\Gamma(n+1)} \sim e^{\theta(1+\theta)}(n+1)^{1+\theta} \qquad \text{as } n \to \infty. \tag{3.33}$$

It follows from (2.9) and (3.33) that

$$c_n^2 = O\left(n^{1+\theta}\right) \qquad \text{as } n \to \infty. \tag{3.34}$$

Hence the condition (3.32) holds and the functional S may be uniquely extended to a bounded linear functional on V, still denoted by $S(r)$, due to Proposition 3.2. We have arrived a crucial point in our approach.

Assumption 3.2. *The nonlinear term $g(W)$ may be replaced by $\chi\varepsilon(\tau)^\gamma S(r)$ with the extension $S(r)$ to V'. Accordingly, the evolution of W is governed by the equation*

$$W_\tau = W_{rr} + \left(\frac{N-1-2\gamma}{r} - \frac{r}{2}\right) W_r + 2W + \chi\varepsilon(\tau)^\gamma S(r) \quad \text{in } V', \tag{3.35}$$

where $\chi = D_{N,2}$.

Notice that the Fourier expansion (3.26) also makes sense in V' since the topology given by the norm of V' is weaker than that of X.

Remark 3.3. It might sound curious to consider the equation in V', since our sought-for solution Φ should belong to $L^2_{r,\rho}(\mathbf{R}^N)$. This is not a contradiction, because we use the *outer solution* only in the region $\{\varepsilon(\tau) \ll r\}$. Therefore the outer solution can grow as $r \to 0$ and do not necessarily belong to $L^2_{r,\rho}(\mathbf{R}^N)$.

Suppose that the order of $Q(r,\tau)$ is $\varepsilon(\tau)^m$ for some $m > 0$, neglecting some possible logarithmic correction such as $\log\varepsilon(\tau), \log|\log\varepsilon(\tau)|$,... Namely,

$$Q(r,\tau) \sim \varepsilon(\tau)^m F(r), \qquad \text{as } \tau \to \infty, \tag{3.36}$$

$$\langle F, \psi_j \rangle = 0 \qquad \text{for } j = 0, 1, 2. \tag{3.37}$$

Since the order of Q should be higher than $a_2(\tau)$ in every compact subset of $(0,\infty)$, we may assume $m > 4$ (cf. (3.11)). It then turns out that

$$\begin{aligned} m\varepsilon(\tau)^{m-1}\dot{\varepsilon}(\tau)F = \varepsilon(\tau)^m \left(F'' + \left(\frac{N-1-2\gamma}{r} - \frac{r}{2}\right) F' + 2F\right) \\ + \chi\varepsilon(\tau)^\gamma \left\{ S - \sum_{k=0}^{2} \langle S, \psi_k \rangle \psi_k \right\} \quad \text{in } V'. \end{aligned} \tag{3.38}$$

Suppose now that $4 < m < \gamma$. Due to the basic hypothesis (3.2) on $\dot{\varepsilon}(\tau)$, the function F should then satisfy

$$F'' + \left(\frac{N-1-2\gamma}{r} - \frac{r}{2}\right) F' + 2F = 0 \quad \text{in } V' \tag{3.39}$$

and (3.37). Similarly to the case of high dimensions (cf. (3.19)), there exists no nontrivial solution of (3.39) in V' satisfying (3.37), whence $F \equiv 0$.

We next suppose that $m = \gamma$. In this case the function F satisfies

$$-\left[F'' + \left(\frac{N-1-2\gamma}{r} - \frac{r}{2}\right) F' + 2F\right] = \chi \left\{ S - \sum_{k=0}^{2} \langle S, \psi_k \rangle \psi_k \right\} \quad \text{in } V'. \tag{3.40}$$

Lemma 3.4. *Assume that $N \le 22$ holds. Then there exists a unique solution of* (3.40) *in V' satisfying* (3.37)*. It is expressed as*

$$F(r) = \sum_{j=3}^{\infty} \frac{1}{j-2} \langle S, \psi_j \rangle \iota \psi_j,$$

where the convergence takes place by the norm of V'. Moreover, there is a positive constant $C(N)$ depending only on N such that

$$F(r) \sim \chi C(N) r^{-(N-2-2\gamma)} \quad \text{as } r \to 0. \tag{3.41}$$

Remark 3.5. The assumption on N in Lemma 3.4 is not optimal. If we consider general dimensions, larger spaces than V' have to be selected in accordance with N.

The proof of Lemma 3.4 is postponed to §4. Loosely speaking, we may understand that the leading term of $F(r)$ as $r \to 0$ is determined by the fundamental solution of the Laplace equation in the "fractional dimension".

We then finally show that the case $m > \gamma$ should be ruled out. Taking a duality product with ψ_n in (3.35), we have

$$\dot{a}_n(\tau) + \lambda_n a_n(\tau) = \chi \varepsilon(\tau)^\gamma \psi_n(0) \tag{3.42}$$

for each n. Since all the Fourier coefficients $a_n(\tau)$, $n \ne 2$, as well as their derivatives should be at most of order $\varepsilon(\tau)^m$, the relation (3.42) implies $\chi\psi_n(0) = 0$, a contradiction.

Hence we conclude $m = \gamma$. It follows from (3.42) again that

$$a_n(\tau) = -\chi \psi_n(0) \int_\tau^\infty e^{-\lambda_n (s-\tau)} \varepsilon(s)^\gamma ds$$

for $n = 0, 1, 2$, whence $|a_0(\tau)| \ll |a_1(\tau)| \ll |a_2(\tau)| \ll 1$ as $\tau \to \infty$. Since $N - 2\gamma - 2 = 2\beta$ and $\gamma - 2\beta = 4$, we then see

$$Q(r,\tau) \sim \chi C(N) \varepsilon(\tau)^\gamma r^{-(N-2\gamma-2)} = o\left(\varepsilon(\tau)^4\right) \quad \text{for } \varepsilon(\tau) \ll r \ll 1$$

as $\tau \to \infty$. Hence the following outer expansions are obtained:

$$\Phi_{\text{out}}(r,\tau) \sim \Phi_\infty(r) + c_2 a_2(\tau) r^{-\gamma} \quad \text{for } \varepsilon(\tau) \ll r \ll 1, \tag{3.43a}$$

$$a_2(\tau) = -\chi c_2 \int_\tau^\infty \varepsilon(s)^\gamma ds. \tag{3.43b}$$

Matching the outer expansions (3.43) with the inner ones (3.7) with $\alpha = 1$ in the intermediate region $\varepsilon(\tau) \ll r \ll 1$, we obtain an asymptotic integral equation

$$-h_1\varepsilon(\tau)^4 \sim -\chi c_2^2 \int_\tau^\infty \varepsilon(s)^\gamma ds.$$

The solution of the corresponding asymptotic differential equation is given by

$$\varepsilon(\tau) \sim (K\tau)^{-(p-1)/2} \quad \text{with} \quad K = \frac{\chi c_2^2}{2(p-1)h_1},$$
$$a_2(\tau) \sim -\frac{12\chi c_2}{N-10} K^{-(N+2)/(N-10)} \tau^{-12/(N-10)},$$

whence $\Phi(0,\tau) \sim K\tau$ as $\tau \to \infty$. Returning to the original variables, we conclude (1.14).

3.3 Threshold dimension case: $N = 22$

Since the principal contributions from I_1 and I_2 are balanced in this case, we shall elaborate the detail. As in the previous cases, we shall show that the neutral mode

$$v(y,\tau) \sim a_2(\tau)\phi_2(r) \tag{3.44}$$

is an only reasonable approximation of the leading term in outer regions. To show this, we return to the computation of I:

$$I = \int_0^\infty f(v(r,\tau))\phi(r) r^{N-1}\rho(r)dr = I_1 + I_2,$$

where ϕ is a continuous function on $\mathbf{R}_+$ such that $\phi(r) = O(r^{-\gamma})$ as $r \to 0$. We then expect that, for r bounded away from zero, v is so small that the nonlinear term $f(v)$ may be approximated by a quadratic function of v. Combining this with (3.9), we obtain

$$I_2 \sim \frac{p(p-1)c_*^{p-2}}{2} B^2\hat{\phi}(0)\varepsilon(\tau)^{2(\gamma-2\beta)} \int_L^\infty r^{-3\gamma-2\beta(p-2)+N-1} e^{-r^2/4} dr,$$

where $\hat{\phi}(0) := \lim_{r\to 0+} r^\gamma \phi(r)$ is a constant. After some algebraic computations, we obtain $-3\gamma - 2\beta(p-2) + N - 1 = -1$ and thus

$$I_2 \sim -\frac{p(p-1)c_*^{p-2}}{2} B^2\hat{\phi}(0)\varepsilon(\tau)^8 \log L \quad \text{as } \tau \to \infty.$$

On the other hand, we have

$$I_1 \sim \frac{p(p-1)}{2} c_*^{p-2} h_\alpha^2 \hat{\phi}(0)\varepsilon(\tau)^8 \log\frac{L}{\varepsilon(\tau)}.$$

Since the whole formula of I doesn't depend on L, the free constant α should be determined so that $h_\alpha = B$. As a result, we obtain

$$\langle f(v(\cdot,\tau)), \phi\rangle \sim E\hat{\phi}(0)\varepsilon(\tau)^8 |\log\varepsilon(\tau)| \quad \text{as } \tau \to \infty$$
$$\text{with } E = \frac{p(p-1)}{2} h_\alpha^2 c_*^{p-2} > 0. \tag{3.45}$$

It is worth pointing out that the Dirac and the Taylor approximations of $f(v)$ are balanced as it can be seen in the appearance of the logarithmic factor. It is fascinating to replace $f(v)$ by the Dirac mass with suitable strength. A better way to see this is to argue with the dependent variable W as in (3.23), which solves equation (3.24). The matching condition (3.9) reads

$$W(r,\tau) \sim -B\varepsilon(\tau)^4 \tag{3.46}$$

as $r \to 0$. Due to the basic assumption that $p = p_{\mathrm{L}}$ with $N = 22$, we have $\gamma = 8$. Equation (3.24) is then reduced to

$$W_\tau = W_{rr} + \left(\frac{5}{r} - \frac{r}{2}\right) W_r + 2W + r^8 f\left(r^{-8} W\right). \tag{3.47}$$

This is essentially a six-dimensional parabolic equation and function W is naturally understood to be a radial solution of

$$W_\tau = \Delta_{(6)} W - \frac{y \cdot \nabla_{(6)} W}{2} + 2W + g(W), \tag{3.48a}$$

$$g(W) := |y|^8 f\left(|y|^{-8} W\right), \tag{3.48b}$$

where $\Delta_{(6)}$ and $\nabla_{(6)}$ stand for the Laplacian and gradient in $\mathbf{R}^6$, respectively. The linear operator defined by

$$-\mathcal{L}\psi = \Delta_{(6)}\psi - \frac{y \cdot \nabla_{(6)}\psi}{2} + 2\psi, \quad \psi \in D(\mathcal{L}) \equiv V \simeq H^2_{r,\rho}(\mathbf{R}^6)$$

is self-adjoint. Its spectrum consists only of the same eigenvalues as of operator A and

$$\mathcal{L}\psi_n = \lambda_n \psi_n, \qquad \phi_n(r) = r^{-8}\psi_n(r)$$

for $n = 0, 1, 2, \ldots$, where ϕ_n is the eigenfunction of A associated with eigenvalue λ_n. We now impose the following assumption instead of Assumption 3.2.

Assumption 3.3. *The nonlinear term $g(W)$ may be replaced by $E\varepsilon(\tau)^8 |\log \varepsilon(\tau)|\, \delta(y)$. Accordingly, the evolution of W is governed by the equation*

$$W_\tau = \Delta_{(6)} W - \frac{y \cdot \nabla_{(6)} W}{2} + 2W + E\varepsilon(\tau)^8 |\log \varepsilon(\tau)|\, \delta(y) \quad \text{in } V', \tag{3.49}$$

where $\delta(y)$ denotes the extension to V' of the Dirac mass supported at the origin of $\mathbf{R}^6$.

As far as the author knows, the approximation of the nonlinear term by means of the Dirac mass was first used in the context of the classical Stephan problem [11].

Let us write

$$W(r,\tau) = a_0(\tau)\psi_0(r) + a_1(\tau)\psi_1(r) + a_2(\tau)\psi_2(r) + Q(r,\tau),$$
$$\langle Q(\cdot,\tau), \psi_j\rangle = 0 \quad (j = 0, 1, 2).$$

Arguing as in §§3.2, we expect the magnitude of $Q(r,\tau)$ as $\tau \to \infty$ is $\varepsilon(\tau)^8 |\log \varepsilon(\tau)|$:

$$Q(y,\tau) \sim E\varepsilon(\tau)^8 |\log \varepsilon(\tau)|\, F(y)$$

as $\tau \to \infty$, where $F \in V'$ is a radial function satisfying

$$
\begin{aligned}
&-\Delta_{(6)}F + \frac{y \cdot \nabla_{(6)}F}{2} - 2F = \delta(y) - \sum_{k=0}^{2} \langle \delta, \psi_k \rangle \psi_k \quad \text{in } V', \\
&\langle F, \psi_j \rangle = 0 \quad (j = 0, 1, 2).
\end{aligned}
$$

It then follows from Lemma 3.4 that the function F behaves like the fundamental solution of the Laplace equation in $\mathbf{R}^6$, that is,

$$
F(y) \sim C_6 |y|^{-4} \quad \text{as } |y| \to 0, \tag{3.50}
$$

where $C_6 > 0$ is a universal constant. Summarizing, we have obtained

$$
Q(y, \tau) \sim C_6 E \varepsilon(\tau)^8 \left|\log \varepsilon(\tau)\right| |y|^{-4}.
$$

This is smaller than $\varepsilon(\tau)^4$ in the intermediate region:

$$
\varepsilon(\tau) |\log \varepsilon(\tau)|^{1/4} \ll |y| \ll 1. \tag{3.51}
$$

Notice that the function Q is no longer subdominant for, e.g., $|y| = \varepsilon(\tau)|\log \varepsilon(\tau)|^{1/8}$ in this case. For this reason, unlike the previous cases, we have to restrict the outer region to $\{\varepsilon(\tau)|\log \varepsilon(\tau)|^{1/4} \ll |y|\}$, in which the Dirac mass contributes to the leading term only through the neutral mode $a_2(\tau)\psi_2(y)$. The matching condition (3.9) is then satisfied with $B = c_2$ (cf. (2.9)). We have obtained the following outer expansions:

$$
\Phi_{\text{out}}(r, \tau) \sim \Phi_\infty(r) + c_2 a_2(\tau) r^{-\gamma} \quad \text{for } \varepsilon(\tau) \ll r \ll 1, \tag{3.52a}
$$

$$
a_2(\tau) = E c_2 \int_\tau^\infty \varepsilon(s)^8 \log \varepsilon(s) ds. \tag{3.52b}
$$

Matching the outer expansions (3.52) with the inner ones (3.7) in the inter mediate region (3.51), we obtain an asymptotic integral equation

$$
-h_\alpha \varepsilon(\tau)^4 \sim E c_2^2 \int_\tau^\infty \varepsilon(s)^8 \log \varepsilon(s) ds
$$

as $\tau \to \infty$. The corresponding asymptotic differential equation is then given by

$$
\dot{\varepsilon}(\tau) \sim \frac{E c_2^2}{4 h_\alpha} \varepsilon(\tau)^5 \log \varepsilon(\tau),
$$

which may be solved asymptotically:

$$
\varepsilon(\tau) \sim (D\tau \log \tau)^{-1/4} \quad \text{with} \quad D = \frac{E c_2^2}{4 h_\alpha}, \tag{3.53}
$$

$$
a_2(\tau) \sim -\frac{c_2 E}{4} \tau \log \tau, \tag{3.54}
$$

whence $\Phi(0, \tau) \sim D\tau \log \tau$ as $\tau \to \infty$. Returning to the original variables, we conclude the desired result (1.15).

4 Proofs of some technical results

In this section we prove some technical results in §3.

Proof of Proposition 3.2. Let T be a bounded linear functional on V. Riesz' representation theorem guarantees the existence of $\Phi_S \in V$ with $\|T\|_{V'} = \|\Phi_T\|_V$ such that

$$\langle T, \varphi\rangle = (\Phi_T, \varphi)_V \qquad \text{for any } \varphi \in V. \tag{4.1}$$

Then we have

$$\Phi_T = \sum_{k=0}^{\infty} (\Phi_T, \Psi_k)_V \, \Psi_k = \sum_{k=0}^{\infty} \langle T, \Psi_k\rangle \Psi_k \quad \text{ in } V. \tag{4.2}$$

Identity (3.31) follows from Parseval identity:

$$\|\Phi_T\|^2 = \sum_{k=0}^{\infty} |\langle T, \Psi_k\rangle|^2 = \sum_{k=0}^{\infty} \frac{|\langle T, \psi_k\rangle|^2}{1+\lambda_k^4}.$$

Due to (3.28), (4.1), and (4.2), we have

$$\langle T, \varphi\rangle = \sum_{k=0}^{\infty} \langle T, \Psi_k\rangle (\Psi_k, \varphi)_V = \sum_{k=0}^{\infty} \langle T, \psi_k\rangle (\psi_k, \varphi)_X.$$

This means that $T = \sum_{k=0}^{\infty} \langle T, \psi_k\rangle \iota\psi_k$ and (3.30) follows.

We shall next consider a linear functional T_0 such that $\sum_{j=0}^{\infty} |\langle T_0, \psi_j\rangle|^2/(1+\lambda_j^4) < +\infty$. Let φ be an element of V. It is readily seen by (3.28) that

$$\left| \sum_{j=0}^{\infty} \langle T_0, \Psi_j\rangle \, (\psi_j, \varphi)_X \right| \le \left(\sum_{j=0}^{\infty} \frac{|\langle T_0, \psi_j\rangle|^2}{1+\lambda_j^4} \right)^{1/2} \left(\sum_{j=0}^{\infty} \left|(\Psi_j, \varphi)_V\right|^2 \right)^{1/2}.$$

Due to the assumption and Parseval identity, the last estimate implies that the linear functional $\tilde{T} := \sum_{j=0}^{\infty} \langle T_0, \psi_j\rangle \iota\psi_j$ satisfies

$$|\langle \tilde{T}, \varphi\rangle| \le C\|\varphi\|_V \qquad \text{for every } \varphi \in V,$$

where $C > 0$ is a constant independent of φ. This shows the boundedness of $\tilde{T}$. Uniqueness of such an extension is a direct consequence from (3.30). The proof is complete. $\square$

Proof of Lemma 3.4. Recall the bounded linear operator $\iota : X \to V'$ defined in (3.29). Any element $x \in V$ is identified with $\iota x \in V'$, so that $V \subset X \subset V'$. We first claim that

The subspace V is dense in V' in the sense that $\overline{\iota V} = V'$.

The proof of this last fact proceeds as in the standard theory of Sobolev spaces. Riesz' representation theorem shows that there exists an isomorphism $\Psi \in \mathcal{L}(V', V)$ such that

$$\langle T, \varphi\rangle = (\Psi T, \varphi)_V \qquad \text{for any } T \in V',\ \varphi \in V. \tag{4.3}$$

Suppose that $\overline{\iota V} \neq V'$. It then follows that $\overline{\Psi(\iota V)} \neq V$, where the closure is taken with respect to the norm of V. This implies that the orthogonal subspace of $\overline{\Psi(\iota V)}$ is nontrivial. Hence there exists $h \in V \setminus \{0\}$ such that

$$0 = (\Psi(\iota g), h)_V = (g, h)_X \quad \text{for every } g \in V,$$

where (4.3) and (3.29) have been used. Since V is dense in X, we conclude $h = 0$. This is a contradiction and the claim follows.

Let $\{f_m\} \subset X$ be a sequence such that $\iota f_m \to S$ as $m \to \infty$ and set

$$T_m := f_m - \sum_{k=0}^{2} (f_m, \psi_k)_X \psi_k$$

for $m = 1, 2, \ldots$ Define a linear operator $\mathcal{L}_*$ in X as

$$\mathcal{L}_* F = -\left(F'' + \left(\frac{N-1-2\gamma}{r} - \frac{r}{2} \right) F' \right) + F, \qquad F \in D(\mathcal{L}_*) = V.$$

A standard argument shows that $\mathcal{L}_*$ is a self-adjoint operator with compact inverse $G := \mathcal{L}_*^{-1}$. Notice that the compactness is a consequence of the finite measure $r^{N-1}\rho(r)dr$ associated with the entire space X. Consider an elliptic equation

$$\mathcal{L}_* F - 3F = T_m.$$

This equation is equivalent to

$$F - 3GF = GT_m. \tag{4.4}$$

Since G inherits self-adjointness from $\mathcal{L}_*$, we have

$$(GT_m, \psi_j)_X = \frac{1}{\lambda_j + 3} (T_m, \psi_j)_X = \frac{1}{j+1} (T_m, \psi_j)_X$$

for every j. In particular, we have $(GT_m, \psi_2)_X = 0$. This last fact shows that the sequence $\{GT_m\}_{m=1}^{\infty}$ lies in the orthogonal complement of the kernel $\mathcal{N}(I - 3G) := \{\varphi \in X;\ \varphi - 3G\varphi = 0\} = \{C\psi_2;\ C \in \mathbf{R}\}$. Riesz–Schauder theory then shows that for each m there exists a solution $F_m \in X$ of (4.4) with the form:

$$F_m = \sum_{j \neq 2} \frac{1}{\lambda_j} (T_m, \psi_j)_X \psi_j = \sum_{j=3}^{\infty} a_j^{(m)} \psi_j \quad \text{in } X, \tag{4.5a}$$

$$a_j^{(m)} := \frac{1}{\lambda_j} (T_m, \psi_j)_X = \frac{1}{j-2} (f_m, \psi_j)_X, \quad j = 3, 4, \ldots \tag{4.5b}$$

Moreover, the solution is unique up to adding the corresponding homogeneous solutions i.e., a constant multiple of ψ_2. Notice that the series (4.5a) converges in the norm of V', since Proposition 3.2 implies

$$\|\iota F_m - \iota F_n\|_{V'}^2 = \sum_{j=3}^{\infty} \frac{|a_j^{(m)} - a_j^{(n)}|^2}{1 + \lambda_j^4} \leq \|\iota f_m - \iota f_n\|_{V'}^2.$$

Sending $m \to \infty$ in (4.5a), we obtain

$$F := \lim_{m\to\infty} \iota F_m = \sum_{j=3}^{\infty} \frac{1}{j-2} \langle S, \psi_j \rangle \iota \psi_j \quad \text{in } V'. \tag{4.6}$$

This $F \in V'$ is a solution of (3.40) satisfying (3.37).

Let us set

$$F_0(r) = E r^{-(N-2\gamma-2)} \quad \text{with } E = \frac{1}{2\nu}. \tag{4.7}$$

We now compute $I_n := \int_0^\infty F_0(r)\psi_n(r) r^{N-2\gamma-1} \rho dr$. Due to (4.7) and (3.25b), we have

$$I_n = E D_n \int_0^\infty L_n^{(\nu)} \left(\frac{r^2}{4}\right) r \rho dr \quad \text{with} \quad D_n := c_n \frac{\Gamma(\nu+1)\, n!}{\Gamma(\nu+n+1)},$$

where $\nu > 0$ is the constant as in (3.25b) and $\rho = \exp(-r^2/4)$. The last integral is explicitly computed by the classical formula:

$$\int_0^\infty L_n^{(\nu)}(x) e^{-x} dx = \frac{\Gamma(\nu+n)}{n!\Gamma(\nu)},$$

which is readily seen by the Rodrigues' formula: $L_n^{(\nu)}(x) = \frac{x^{-\nu}e^x}{n!} \frac{d^n}{dx^n}(e^{-x} x^{n+\nu})$. Summarizing, we obtain

$$I_n = \frac{2\nu E}{\nu+n} c_n. \quad \text{for } n = 0, 1, ... \tag{4.8}$$

This is a bounded sequence for $N \leq 22$ due to (3.34), which implies in particular that the function F_0 defines a bounded linear functional on V, still denoted by F_0. It follows that

$$F - \sum_{n=3}^{\infty} \langle F_0, \psi_n \rangle \iota \psi_n = (\nu+2) \sum_{n=3}^{\infty} \frac{c_n}{(n+\nu)(n-2)} \iota \psi_n \tag{4.9}$$

Since (3.34) implies

$$\left(\frac{c_n}{(n+\nu)(n-2)}\right)^2 = O\left(\frac{1}{n^2}\right)$$

the series $\sum_{n=3}^{\infty} \frac{c_n}{(n+\nu)(n-2)} \psi_n$ in fact belongs to X. More precisely, we can argue as above to see

$$\int_0^\infty r^{-(N-2\gamma-2)+2} \psi_n(r) r^{N-2\gamma-1} \rho dr = \frac{C c_n}{(\nu+n)(\nu+n-1)}, \tag{4.10}$$

where $C > 0$ is a constant independent of n. Due to (4.9) and (4.10), the functional F is identified by a function $F(r)$ and

$$F(r) \sim E r^{-(N-2\gamma-2)} \left(1 + O\left(r^2\right)\right) \quad \text{as } r \to 0.$$

The proof is now complete. $\square$

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References

[1] P. Biernat, *Non-self-similar blow-up in the heat flow for harmonic maps in higher dimensions*, Nonlinearity **28** (2015), 167–185.

[2] S. Filippas, M. A. Herrero, and J. J. L. Velázquez, *Fast blow-up mechanisms for sign-changing solutions of a semilinear parabolic equation with critical nonlinearity*, R. Soc. Lond. Proc. Ser. A. **456** (2000), 2957–2982.

[3] S. Filippas and R. V. Kohn, *Refined asymptotics for the blowup of $u_t - \Delta u = u^p$*, Comm. Pure Appl. Math. **45** (1992), 821–869.

[4] Y. Giga and R. V. Kohn, *Asymptotically self-similar blow-up of semilinear heat equations*, Comm. Pure Appl. Math. **38** (1985), 297–319.

[5] ———, *Characterizing blowup using similarity variables*, Indiana Univ. Math. J. **36** (1987), 1–40.

[6] ———, *Nondegeneracy of blowup for semilinear heat equations*, Comm. Pure Appl. Math. **42** (1989), 845–884.

[7] Y. Giga, S. Matsui, and S. Sasayama, *Blow up rate for semilinear heat equations with subcritical nonlinearity*, Indiana Univ. Math. J. **53** (2004), 483–514.

[8] M. A. Herrero and J. J. L. Velázquez, *Blow-up behaviour of one-dimensional semilinear parabolic equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire **10** (1993), 131–189.

[9] ———, *A blow up result for semilinear heat equations in the supercritical case*, unpublished preprint.

[10] ———, *Explosion de solutions d'équations paraboliques semilinéaires supercritiques*, C. R. Acad. Sci. Paris Sér. I Math. **319** (1994), 141–145.

[11] ———, *On the melting of ice balls*, SIAM J. Math. Anal. **28** (1997), 1–32.

[12] Y. Li, *Asymptotic behavior of positive solutions of equation $\Delta u + K(x)u^p = 0$ in $\mathbb{R}^n$*, J. Differential Equations **95** (1992), 304–330.

[13] H. Matano, *Blow-up in nonlinear heat equations with supercritical power nonlinearity*, Contemp. Math., vol. 446, Amer. Math. Soc., Providence, RI, 2007, pp. 385–412.

[14] H. Matano and F. Merle, *On nonexistence of type II blowup for a supercritical nonlinear heat equation*, Comm. Pure Appl. Math. **57** (2004), 1494–1541.

[15] ———, *Classification of type I and type II behaviors for a supercritical nonlinear heat equation*, J. Funct. Anal. **256** (2009), 992–1064.

[16] ———, *Threshold and generic type I behaviors for a supercritical nonlinear heat equation*, J. Funct. Anal. **261** (2011), 717–748.

[17] N. Mizoguchi, *Nonexistence of type II blow-up solution for a semilinear heat equation*, J. Differential Equations **250** (2011), 26–32.

[18] ———, *Blow-up rate of type II and the braid group theory*, Trans. Amer. Math. Soc. **363** (2011), 1419–1443.

[19] P. Quittner and P. Souplet, *Superlinear parabolic problems. Blow-up, global existence and steady states*, Birkhäuser Advanced Texts, Basel, 2007.

[20] Y. Seki, *Type II blow-up mechanisms in a semilinear heat equation with critical Joseph–Lundgren exponent*, preprint.

[21] R. Schweyer, *Type II blow-up for the four dimensional energy critical semi linear heat equation*, J. Funct. Anal. **12** (2012), 3922–3983.