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A duality theorem for a three-phase partition problem

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A duality theorem for a three-phase partition problem. [1]

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Abstract. In some nonlinear diffusive phenomena, the systems have three or more stable states. Sternberg and Zeimer (Ref. 1) established the existence of local minimizers to the problem of partitioning certain domain $\Omega \subset \mathbb{R}^2$ into three subdomains having least interfacial area. Ikota and Yanagida investigated stability and instability for stationary curves with one triple junction in (Ref. 2) and for stationary binary-tree type interfaces in (Ref. 3). In this paper, we consider a static version of the partitioning problem with a triple junction and present a duality theorem. The novelity of our duality theorem is tha it is based on separation of three convex sets by a triangle.

Key words. Duality theorem, Separation, Convex set, Partition problem

1. Introduction

In some nonlinear diffusive phenomena, e.g., grain growth in annealing pure metal and segregation between biological species, the systems have three or more stable states (Fig.1). Sternberg and Zeimer(Ref. 1) established the existence of local minimizers to the problem of partitioning certain domain $\Omega \subset \mathbb{R}^2$ into three subdomains having least interfacial area. Furthermore, Ikota and Yanagida (Refs. 2, 3) investigated stability for stationary curves with one triple junction and of binary-tree type (Fig. 2).

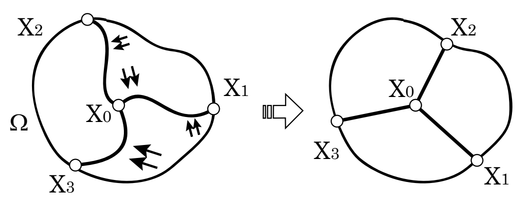

Figure 1. Three-phase partition problem

Although they formulated partitioning problems as variational problems, they can be formulated as extremal problems in $\mathbb{R}^n$, since the shortest curve joining X_0 and X_i is the line segment X_0X_i. From this point of view, the author discussed stability and instability of the

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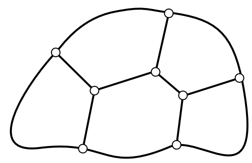

FIGURE 2. Binary-tree type interface

three-phase partition problem and studied its game-theoretic aspect in Kawasaki (Refs. 4, 5).

In this paper, we formulate the three-phase partition problem as follows. Let C_i $(i = 1, 2, 3)$ be closed convex sets with non-empty interior in $\mathbb{R}^2$ such that $\Omega := \cap_{i=1}^{3} C_i^c$ is non-empty (Fig. 3).

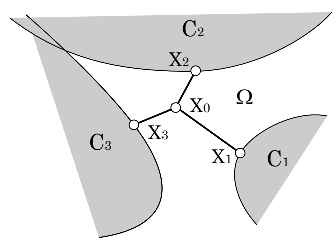

FIGURE 3. Primal problem

$$(P)\qquad \begin{array}{ll} \text{Minimize} & f(X_0, \ldots, X_3) := \displaystyle\sum_{i=1}^{3} ||X_i - X_0|| \\ \text{subject to} & X_i \in C_i \ (i = 1, 2, 3) \\ & X_0 \in \Omega. \end{array}$$

When we emphasize the domain Ω, we denote (P) by (P_Ω). Although Ω is not convex, primal problem (P) can be regarded as a convex programming problem if X_0 is restricted to an open convex subset C_0 of Ω. The aims of this paper are to give a dual problem (D) and show duality between (P) and (D).

This paper is organized as follows. In Section 2, we give first-order optimality conditions for (P). In Section 3, we briefly review classical duality theorems and introduce a new concept of separation of three convex sets by a triangle. In Section 4, we define the dual problem (D) and show duality.

2. First-order optimality condition

In this section, we first give a first-order necessary optimality condition for (P). Next, we consider the special case that C_i's are closed half spaces.

A local minimizer $(X_0, \ldots, X_3)$ of (P) is said to be non-degenerate if X_0 does not coincide with any X_i $(i = 1, 2, 3)$.

Let $N(X_i; C_i)$ denote the normal cone of C_i at X_i, that is,

$$N(X_i; C_i) := \{\xi \in \mathbb{R}^2 \mid \xi^T(X - X_i) \leq 0 \quad \forall X \in C_i\}.$$

Theorem 2.1. *Let $(X_0, \ldots, X_3)$ be a non-degenerate minimal solution for (P), Then it satisfies Young's law*

$$\angle X_i X_0 X_j = 120^\circ \text{ for any } i \neq j$$

and the transversality condition

$$X_0 - X_i \in N(X_i; C_i) \quad (i = 1, 2, 3).$$

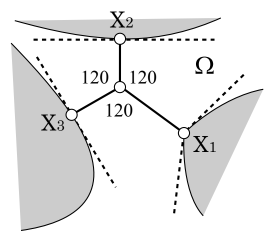

FIGURE 4. Young's law and the transversality condition

Proof. According to Kuhn-Tucker's theorem, see e.g. Rockafellar (Ref. 6, Section 28), there exist Kuhn-Tucker multipliers $\lambda_i \geq 0$ $(i = 1, 2, 3)$ such that $0 \in \mathbb{R}^8$ belongs to the subdifferential of the Lagrange function

$$L(X_0, \ldots, X_3) := f(X_0, \ldots, X_3) + \sum_{i=1}^{3} \lambda_i \delta(X_i | C_i),$$

where $\delta(X_i|C_i)$ denotes the characteristic function of C_i. Picking up X_0-component of the subdifferential ∂L, we have

$$\sum_{i=1}^{3} \frac{X_i - X_0}{||X_i - X_0||} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{1}$$

Denoting

$$e_i := (X_i - X_0)/||X_i - X_0||, \tag{2}$$

we get from (1) that $||e_k||^2 = ||e_i||^2 + ||e_j||^2 + 2e_i^T e_j$, where $\{i, j, k\} = \{1, 2, 3\}$. Hence $e_i^T e_j = -1/2$, which implies Young's law (Fig. 4).

Picking up X_i-component $(i = 1, 2, 3)$ of ∂L, we have

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \frac{X_i - X_0}{||X_i - X_0||} + \lambda_i N(X_i; C_i). \tag{3}$$

Hence we get the transversality condition. □

Next, we consider the special case that each C_i is a closed half space defined by

$$C_i = \{X_i \mid \xi_i^T X_i \le \alpha_i\}, \tag{4}$$

where ξ_i is assumed to be a unit vector. Then (P) becomes a convex programming problem

$$(P) \qquad \begin{array}{ll} \text{Minimize} & \displaystyle\sum_{i=1}^{3} ||X_i - X_0|| \\ \text{subject to} & \xi_i^T X_i \le \alpha_i \ (i = 1, 2, 3) \\ & \xi_i^T X_0 \ge \alpha_i \ (i = 1, 2, 3), \end{array}$$

and optimal solutions are characterized by the first-order optimality condition. Furthermore, $N(X_i; C_i) = \{\lambda_i \ge 0 \mid \lambda_i \xi_i\}$ at $X_i \in \partial C_i$. Hence, if $(X_0, \ldots, X_3)$ satisfies both Young's law and the transversality condition, then Ω must be a regular triangle (Fig. 5). Hence we have

$$\min(P) = \Omega\text{'s height.} \tag{5}$$

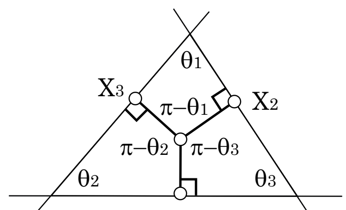

FIGURE 5. Young's law and the transversality condition are satisfied only when $\theta_1 = \theta_2 = \theta_3$.

Otherwise, for any minimum solution $(X_0, \ldots, X_3)$, X_0 must be on the boundary of Ω.

Proposition 2.1. *Assume that C_i's are defined by (4). If $(X_0, \ldots, X_3)$ satisfying $X_0 = X_1 \notin \{X_2, X_3\}$ is a minimum solution for (P), then $\angle X_2 X_0 X_3 \ge 2\pi/3$, the normal vector ξ_1 equally divides angle $\angle X_2 X_0 X_3$, and line segment $X_0 X_i$ $(i = 2, 3)$ orthogonally intersects ∂C_i, respectively. Furthermore, Ω is an isosceles triangle and*

$$\min(P) = \textit{the smaller height of } \Omega. \tag{6}$$

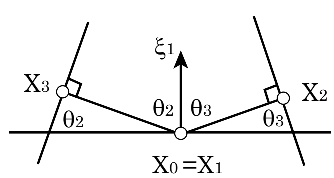

FIGURE 6. Degenerate case: $X_0 = X_1 \ne X_2$, X_3

Proof. There exist $\lambda_i \geq 0$ $(i = 1, 2, 3)$ and $\mu_1 \geq 0$ such that $0 \in \mathbb{R}^8$ belongs to the subdifferential of the Lagrange function

$$L(X_0, \ldots, X_3) := \sum_{i=1}^{3} ||X_i - X_0|| + \sum_{i=1}^{3} \lambda_i(\xi_i^T X_i - \alpha_i) + \mu_1(\alpha_1 - \xi_1^T X_0). \tag{7}$$

We regard $||X_1 - X_0||$ as a function of not (X_0, X_1) but $\boldsymbol{X} := (X_0, \ldots, X_3)$, and denote it by $f_1(X_0, \ldots, X_3)$. Then it follows from (Ref. 6, Theorem 23.9) that its subdifferential at $X_0 = X_1$ w.r.t. $\boldsymbol{X}$ is given by

$$\partial f_1(\boldsymbol{X}) = \left\{ (s,t) \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \;\middle|\; s^2 + t^2 \leq 1 \right\} \tag{8}$$

$$= \left\{ (-s, -t, s, t, 0, 0, 0, 0) \mid s^2 + t^2 \leq 1 \right\}. \tag{9}$$

So there exists (s, t) such that $s^2 + t^2 \leq 1$,

$$(X_0 - \text{component}) \qquad (s,t) + e_2 + e_3 + \mu_1 \xi_1 = (0,0), \tag{10}$$

$$(X_1 - \text{component}) \qquad (s,t) + \lambda_1 \xi_1 = (0,0), \tag{11}$$

$$(X_i - \text{component}) \qquad e_i + \lambda_i \xi_i = (0,0) \quad (i = 2, 3), \tag{12}$$

where $e_i = (X_i - X_0)/||X_i - X_0||$. (12) implies the transversality condition $e_i = -\xi_i$ at X_i. We see from (11) that $0 \leq \lambda_1 \leq 1$. Substituting (11) into (10), we have

$$e_2 + e_3 = (\lambda_1 - \mu_1)\xi_1. \tag{13}$$

We see from (13) that ξ_1 equally divides angle $\angle X_2 X_0 X_3$ and $\lambda_1 - \mu_1 \geq 0$. So, $||e_2 + e_3|| = ||(\lambda_1 - \mu_1)\xi_1|| \leq 1$. Hence $e_2^T e_3 \leq -1/2$. It follows from the transversality condition that Ω has to be an isosceles triangle (Fig. 6). Since θ_2 in Fig. 6 is greater than or equal to $\pi/3$, $\min(P)$ is equal to the smaller height of the isosceles triangle. □

Finally, we consider the second degenerate case: $X_0 = X_1 = X_2 \neq X_3$.

Proposition 2.2. *Assume that C_i's are defined by (4). If $(X_0, \ldots, X_3)$ satisfying $X_0 = X_1 = X_2 \neq X_3$ is a minimum solution for (P), then line segment X_0X_3 orthogonally intersects ∂C_3 and X_0X_3 is the smallest height of the triangle. Hence*

$$\min(P) = \textit{the smallest height of } \Omega. \tag{14}$$

Proof. There exist $\lambda_i \geq 0$ $(i = 1, 2, 3)$ and $\mu_i \geq 0$ $(i = 1, 2)$ such that $0 \in \mathbb{R}^8$ belongs to the subdifferential of the Lagrange function

$$L(X_0, \ldots, X_3) := \sum_{i=1}^{3} ||X_i - X_0|| + \sum_{i=1}^{3} \lambda_i(\xi_i^T X_i - \alpha_i) + \sum_{i=1}^{2} \mu_i(\alpha_i - \xi_i^T X_0). \tag{15}$$

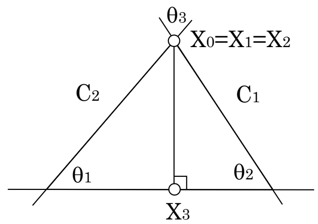

FIGURE 7. Degenerate case: $X_0 = X_1 = X_2 \neq X_3$

Regarding $||X_2 - X_0||$ as a function of $\boldsymbol{X} = (X_0, \ldots, X_3)$, we denote it by $f_2(X_0, \ldots, X_3)$. Then it follows from (Ref. 6, Theorem 23.9) that its subdifferential at $X_0 = X_1$ w.r.t. $\boldsymbol{X}$ is given by

$$\partial f_2(\boldsymbol{X}) = \left\{(-u, -v, 0, 0, u, v, 0, 0) \mid u^2 + v^2 \leq 1\right\}. \tag{16}$$

So there exist (s, t) and (u, v) such that $s^2 + t^2 \leq 1$, $u^2 + v^2 \leq 1$,

$$(X_0 - \text{component}) \qquad (s,t) + (u,v) + e_3 + \mu_1\xi_1 + \mu_2\xi_2 = (0,0), \tag{17}$$
$$(X_1 - \text{component}) \qquad (s,t) + \lambda_1\xi_1 = (0,0), \tag{18}$$
$$(X_2 - \text{component}) \qquad (u,v) + \lambda_2\xi_2 = (0,0), \tag{19}$$
$$(X_3 - \text{component}) \qquad e_3 + \lambda_3\xi_3 = (0,0). \tag{20}$$

(20) implies the transversality condition $e_3 = -\xi_3$ at X_3. We see from (18) that $0 \leq \lambda_1 \leq 1$. Substituting (18), (19), and (20) into (17), we have

$$(\lambda_1 - \mu_1)\xi_1 + (\lambda_2 - \mu_2)\xi_2 + \xi_3 = 0. \tag{21}$$

Since each ξ_i is a normal vector of the half space C_i and since C_i's form a triangle (Fig. 8), it is easily seen that $\sin\theta_i > 0$ and

$$\sin\theta_1\xi_1 + \sin\theta_2\xi_2 + \sin\theta_3\xi_3 = 0. \tag{22}$$

Hence we see from (21) and (22) that $0 \leq \lambda_i - \mu_i \leq 1$, $(i = 1, 2)$. It is clear from Fig. 8 that $\xi_i^T\xi_j = -\cos\theta_k$, where $\{i, j, k\} = \{1, 2, 3\}$. Hence, from (21), we get

$$\lambda_1 - \mu_1 - (\lambda_2 - \mu_2)\cos\theta_3 - \cos\theta_2 = 0. \tag{23}$$

$$-(\lambda_1 - \mu_1)\cos\theta_3 + \lambda_2 - \mu_2 - \cos\theta_1 = 0. \tag{24}$$

It follows from (23) and (24) that $\sin\theta_2 = (\lambda_2 - \mu_2)\sin\theta_3 \leq \sin\theta_3$. Similarly, we have $\sin\theta_1 \leq \sin\theta_3$. Since $\theta_1 + \theta_2 + \theta_3 = \pi$, we see that θ_3 is the largest angle. So X_0X_3 is the smallest height of the triangle. □

Unifying (5), (6), and (14), we concludes that

Corollary 2.1. *When Ω is a triangle,* $\min(P)$ *equals to the smallest height of* Ω*.*

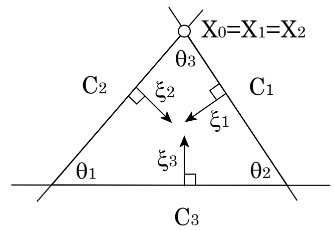

FIGURE 8. Normal vectors of the triangle

3. CLASSICAL DUALITY THEOREMS AND SEPARATION BY A TRIANGLE

In this section, we first review classical duality theorems in brief. Next, we introduce a new concept of separating three convex sets by a triangle. For the sake of simplicity, we choose $\mathbb{R}^2$ as the stage.

On of the simplest duality theorem is the following. Let C_1 be a non-empty convex set in $\mathbb{R}^2$ and $A \notin C_1$ a point. Then the primal problem is

$$(P_1) \qquad \begin{array}{ll} \text{Minimize} & ||X_1 - A|| \\ \text{subject to} & X_1 \in C_1, \end{array}$$

where $||\cdot||$ denotes the Euclidean norm.

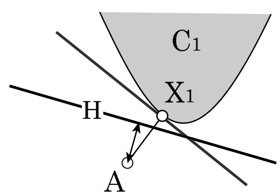

FIGURE 9. Dual problem (D_1)

Its dual problem is to maximize the distance from A to a hyperplane that separates A and C_1 (Fig. 9).

$$(D_1) \qquad \begin{array}{ll} \text{Maximize} & d(A, H) \\ \text{subject to} & H \text{ separates } A \text{ and } C_1, \end{array}$$

where $d(A, H) := \min\{||X - A|| \mid X \in H\}$.

We can rephrase the dual problem as maximizing the width of a strip that separates A and C_1 (Fig. 10).

If we replace A by a convex set C_2 such that $C_1 \cap C_2 =$, then the primal problem is as follows.

$$(P_2) \qquad \begin{array}{ll} \text{Minimize} & ||X_1 - X_2|| \\ \text{subject to} & X_i \in C_i \ (i = 1, 2). \end{array}$$

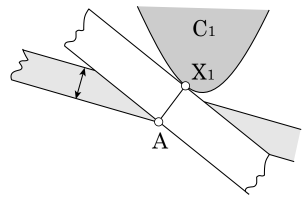

FIGURE 10. Another expression of (D_1)

Its dual problem (D_2) is to minimize the width of a strip that separates C_1 and C_2 (Fig. 11).

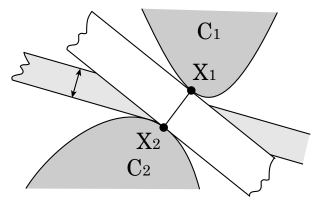

FIGURE 11. Dual problem (D_2)

If we take the epigraph of a convex fucntion f and the hypograph of a concave function g as C_1 and C_2, respectively, and measure the width of the strip in the vertical direction, duality between (P_2) and (D_2) reduces to Fenchel's duality (Fig. 11), see e.g. (Ref. 6, Theorem 31.1).

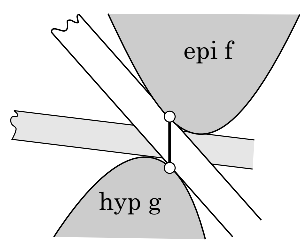

FIGURE 12. Fenchel's duality

$$(P_F) \qquad \text{Minimize} \quad f(x) - g(x),$$

$$(D_F) \qquad \text{Maximize} \quad g_*(y) - f^*(y),$$

where $f^*(y) := \sup_x\{y^T x - f(x)\}$ and $g_*(y) := \inf_y\{y^T x - g(x)\}$.

Definition 3.1. Let C_i $(i = 1, 2, 3)$ be convex sets in $\mathbb{R}^2$ such that $\Omega = \cap_{i=1}^3 C_i^c$ is not empty, and $\Delta \subset \Omega$ a triangle. Then, we say that Δ separates $\{C_i\}_{i=1}^3$ if there are three closed half spaces $\{H_i^-\}_{i=1}^3$ such that $C_i \subset H_i^-$ for every i and $\Delta = \cap_{i=1}^3 H_i^+$, where H_i^+ denotes the closed half space opposite to H_i^- (Fig. 13).

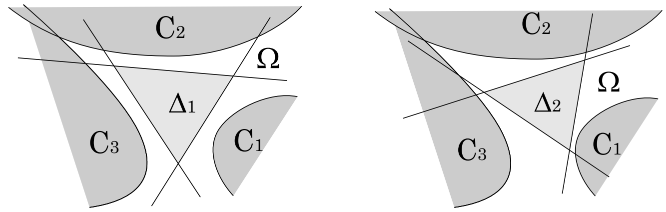

FIGURE 13. Δ_1 separates C_i's and Δ_2 does not separate C_i's.

Lemma 3.1. *Let $(X_0, \dots, X_3)$ be a minimum solution for (P_Ω) and let a triangle Δ separate $\{C_i\}_{i=1}^3$. If X_0 belongs to Δ, then $\min(P_\Delta) \leq \min(P_\Omega)$.*

Proof. Since $X_i \in C_i \subset H_i^-$, we have

$$\min(P_\Omega) = \sum_{i=1}^{3} ||X_i - X_0|| \geq \sum_{i=1}^{3} d(X_0, H_i^-) \geq \min(P_\Delta).$$

□

4. Duality theorem

The following (D) is our dual problem. When Ω is bounded, it has a simplified form (D^*).

	Maximize	the smallest height of a triangle Δ
(D)	subject to	there exists a triangle Δ' such that $\Delta \subset \Delta' \subset \Omega$,
		Δ' separates $\{C_i\}_{i=1}^3$, and $X_0 \in \Delta'$.

Theorem 4.1. *If (X_0, X_1, X_2, X_3) is a non-degenerate mimimum for (P_Ω), then*

$$\min(P_\Omega) = \max(D). \tag{25}$$

Proof. Combining Lemma 3.1 and Corollary 2.1, we have

$$\min(P_\Omega) \geq \min(P_{\Delta'}) \geq \text{the smallest height of } \Delta. \tag{26}$$

By Theorem 2.1, the non-degenerate minimum solution forms a regular triangle Δ^* such that

$$\min(P_\Omega) = \text{the smallest height of } \Delta^*. \tag{27}$$

It follows from definition of the normal cone that Δ^* itself separates $\{C_i\}_{i=1}^3$. Therefore, Δ^* attains the maximum of (D). □

Theorem 4.2. *When Ω is bounded, the dual problem (D) is simplified as follows.*

(D^*)	*Maximize*	*the smallest height of a triangle Δ*
	subject to	$X_0 \in \Delta \subset \Omega$.

Proof. Assume that $\Delta \subset \Omega$. Then, by separation theorem, there are closed half spaces H_i^- $(i = 1, 2, 3)$ such that $C_i \subset H_i^-$ and $\Delta \subset \cap_{i=1}^3 (H_i^+)^c =: \Delta'$. Since Ω is bounded and since $\Delta' \subset \cap_{i=1}^3 C_i^c = \Omega$, Δ' is a triangle separating C_i $(i = 1, 2, 3)$, see Fig. 14. □

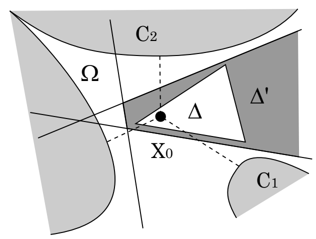

FIGURE 14. Separating hyperplanes form an unbounded polygon.

5. Concluding remark

When Ω is not bounded, separating hyperplanes do not necessarily form a triangle, see Fig. 14. So duality relation $\min(P) = \max(D^*)$ does not always hold. Indeed, since we can enlarge Δ rightward within the dark gray area as we like, $\sup(D^*)$ equals $+\infty$.

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