Lifting Galois representations over arbitrary number fields

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Abstract

It is proved that every two-dimensional residual Galois representation of the absolute Galois group of an arbitrary number field lifts to a characteristic zero p-adic representation, if local lifting problems at places above p are unobstructed.

1 Introduction

Let $\mathbf{k}$ be a finite field of characteristic $p \geq 3$. Let K be a number field of finite degree over $\mathbb{Q}$ and G_K its absolute Galois group $\mathrm{Gal}(\bar{K}/K)$. We consider continuous representations

$$\bar{\rho} : G_K \to \mathrm{GL}_2(\mathbf{k}).$$

The central question that we study in this paper is the existence of a lift of $\bar{\rho}$ to $W(\mathbf{k})$, the ring of Witt vectors of $\mathbf{k}$. This question has been motivated by a conjecture of Serre ([S1]), that is, all odd absolutely irreducible continuous representations $\bar{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbf{k})$ are modular of prescribed weight, level and character. This predicts the existence of a lift to characteristic zero. This conjecture was proved by Khare and Wintenberger in [KW1,KW2]. In [K], Khare proved the existence of lifts to $W(\mathbf{k})$ for any $\bar{\rho} : G_K \to \mathrm{GL}_2(\mathbf{k})$ which are reducible. Ramakrishna proved under very general conditions on $\bar{\rho}$ that there exist lifts to $W(\mathbf{k})$ for $K = \mathbb{Q}$ in [R1,R2]. Gee's results ([G]) imply that there exist lifts to $W(\mathbf{k})$ for $p \geq 5$ and K satisfying $[K(\mu_p) : K] \geq 3$, where μ_p is the group of p-th roots of unity. Böckle and Khare have proved the general n-dimensional case for function field in [BK]. In this paper, we extend Theorem 1 of [R1] to arbitrary number fields. In particular, we will omit the condition $[K(\mu_p) : K] \geq 3$. Hence we can take the field K to be $\mathbb{Q}(\mu_p)^+$, the totally real subfield of $\mathbb{Q}(\mu_p)$.

For a place v of K, let K_v be the completion of K at v, and let G_v be its absolute Galois group $\mathrm{Gal}(\bar{K}_v/K_v)$. Let $\mathrm{Ad}^0\,\bar{\rho}$ be the set of all trace zero two-by-two matrices over $\mathbf{k}$ with Galois action through $\bar{\rho}$ by conjugation. Our main result is the following:

Theorem. *Let K be a number field, and let $\bar{\rho} : G_K \to \mathrm{GL}_2(\mathbf{k})$ be a continuous representation with coefficients in a finite field $\mathbf{k}$ of characteristic $p \geq 7$. Assume that $H^2(G_v, \mathrm{Ad}^0\,\bar{\rho}) = 0$ for each places $v \mid p$. Then $\bar{\rho}$ lifts to a continuous* *representation $\rho : G_K \to \mathrm{GL}_2(W(\mathbf{k}))$ which is unramified outside a finite set of places of K.*

Our method used in the proof is essentially that of Ramakrishna [R1,R2]. In this paper, we follow the more axiomatic treatment presented in [T]. In Section 2, we recall a criterion of Ramakrishna [R2] and Taylor [T] for lifting problems. In Section 3, we define good local lifting problems at certain unramified places and ramified places not dividing p, which will be used in Section 4. In Section 4, we prove Theorem by using the criterion in Section 2 and local lifting problems in Section 3.

Throughout this paper, we assume that p is a prime ≥ 7.

2 A criterion for lifting problems

In this section we recall a criterion of Ramakrishna [R2] and Taylor [T] for a lifting from a fixed residual Galois representation to a p-adic Galois representation.

Let $\mathbf{k}$ be a finite field of characteristic p. Throughout this paper, we consider a continuous representation

$$\bar{\rho} : G_K \to \mathrm{GL}_2(\mathbf{k}).$$

Let S denote a finite set of places of K containing the places above p, the infinite places and the places at which $\bar{\rho}$ is ramified, and let K_S denote the maximal algebraic extension of K unramified outside S. Thus $\bar{\rho}$ factors through $\mathrm{Gal}(K_S/K)$. Put $G_{K,S} = \mathrm{Gal}(K_S/K)$. For each place v of K, we fix an embedding $\bar{K} \subset \bar{K}_v$. This gives a corresponding continuous homomorphism $G_v \to G_{K,S}$.

Let $\mathcal{A}$ be the category of complete noetherian local rings $(R, \mathfrak{m}_R)$ with residue field $\mathbf{k}$ where the morphisms are homomorphisms that induce the identity map on the residue field.

Fix a continuous homomorphism $\delta : G_{K,S} \to W(\mathbf{k})^\times$, and for every $(R, \mathfrak{m}_R) \in \mathcal{A}$ let δ_R be the composition $\delta_R : G_{K,S} \to W(\mathbf{k})^\times \to R^\times$. Suppose $\bar{\rho} : G_{K,S} \to \mathrm{GL}_2(\mathbf{k})$ has $\det \bar{\rho} = \delta_{\mathbf{k}}$.

By a *δ-lift* (resp. *$\delta|_{G_v}$-lift*) of $\bar{\rho}$ (resp. $\bar{\rho}|_{G_v}$) we mean a continuous representation $\rho : G_{K,S} \to \mathrm{GL}_2(R)$ (resp. $\rho_v : G_v \to \mathrm{GL}_2(R)$) for some $(R, \mathfrak{m}_R) \in \mathcal{A}$ such that $\rho \pmod{\mathfrak{m}_R} = \bar{\rho}$ (resp. $\rho_v \pmod{\mathfrak{m}_R} = \bar{\rho}|_{G_v}$) and $\det\rho = \delta_R$ (resp. $\det\rho_v = \delta_R|_{G_v}$). Let $\mathrm{Ad}^0\,\bar{\rho}$ be the set of all trace zero two-by-two matrices over $\mathbf{k}$ with Galois action through $\bar{\rho}$ by conjugation.

Definition 1. For a place v of K, we say that a pair $(\mathcal{C}_v, L_v)$, where $\mathcal{C}_v$ is a collection of $\delta|_{G_v}$-lifts of $\bar{\rho}|_{G_v}$ and L_v is a subspace of $H^1(G_v, \mathrm{Ad}^0\,\bar{\rho})$, is *locally admissible* if it satisfies the following conditions:

(P1) $(\mathbf{k}, \bar{\rho}|_{G_v}) \in \mathcal{C}_v$.

(P2) The set of $\delta|_{G_v}$-lifts in $\mathcal{C}_v$ to a fixed ring $(R, \mathfrak{m}_R) \in \mathcal{A}$ is closed under conjugation by elements of $1 + \mathrm{M}_2(\mathfrak{m}_R)$.

(P3) If $(R, \rho) \in \mathcal{C}_v$ and $f : R \to S$ is a morphism in $\mathcal{A}$ then $(S, f \circ \rho) \in \mathcal{C}_v$.

(P4) Suppose that (R_1, ρ_1) and $(R_2, \rho_2) \in \mathcal{C}_v$, and I_1 (resp. I_2) is an ideal of R_1 (resp. R_2) and that $\phi : R_1/I_1 \xrightarrow{\sim} R_2/I_2$ is an isomorphism such that $\phi\,(\rho_1 \pmod{I_1}) = \rho_2 \pmod{I_2}$. Let R_3 be the fiber product of R_1 and R_2 over $R_1/I_1 \xrightarrow{\sim} R_2/I_2$. Then $(R_3, \rho_1 \oplus \rho_2) \in \mathcal{C}_v$.

(P5) If $((R, \mathfrak{m}_R), \rho)$ is a $\delta|_{G_v}$-lift of $\bar{\rho}|_{G_v}$ such that each $(R/\mathfrak{m}_R^n, \rho \pmod{\mathfrak{m}_R^n}) \in \mathcal{C}_v$ then $(R, \rho) \in \mathcal{C}_v$.

(P6) For $(R, \mathfrak{m}_R) \in \mathcal{A}$, suppose that I is an ideal of R with $\mathfrak{m}_R I = (0)$. If $(R/I, \rho) \in \mathcal{C}_v$ then there is a $\delta|_{G_v}$-lift $\tilde{\rho}$ of $\bar{\rho}|_{G_v}$ to R such that $(R, \tilde{\rho}) \in \mathcal{C}_v$ and $\tilde{\rho} \pmod{I} = \rho$.

(P7) Suppose that $((R, \mathfrak{m}_R), \rho_1)$ and (R, ρ_2) are $\delta|_{G_v}$-lifts of $\bar{\rho}$ with $(R, \rho_1) \in \mathcal{C}_v$, and that I is an ideal of R with $\mathfrak{m}_R I = (0)$ and $\rho_1 \pmod{I} = \rho_2 \pmod{I}$. We shall denote by $[\rho_2 - \rho_1]$ an element of $H^1(G_v, \mathrm{Ad}^0\,\bar{\rho}) \otimes_{\mathbf{k}} I$ defined by $\sigma \mapsto \rho_2(\sigma)\rho_1(\sigma)^{-1} - 1$. Then $[\rho_2 - \rho_1] \in L_v \otimes_{\mathbf{k}} I$ if and only if $(R, \rho_2) \in \mathcal{C}_v$.

Remark 1. Note that we do regard $\mathcal{C}_v$ as a functor from $\mathcal{A}$ to the category of sets.

Let S_{f} be the subset of S consisting of finite places. Throughout this section, suppose that for each $v \in S_{\mathrm{f}}$ a locally admissible pair $(\mathcal{C}_v, L_v)$ is given.

Let $\bar{\chi}_p : G_K \to \mathbf{k}^\times$ be the mod p cyclotomic character. For the $\mathbf{k}[G_K]$-module $\mathrm{Ad}^0\,\bar{\rho}$, by $\mathrm{Ad}^0\,\bar{\rho}(i)$ for $i \in \mathbb{Z}$ we denote the twist of $\mathrm{Ad}^0\,\bar{\rho}$ by the ith tensor power of $\bar{\chi}_p$, and by $\mathrm{Ad}^0\,\bar{\rho}^* := \mathrm{Hom}(\mathrm{Ad}^0\,\bar{\rho}, \mathbf{k})$ we denote its dual representation. The G_K-equivariant trace pairing $\mathrm{Ad}^0\,\bar{\rho} \times \mathrm{Ad}^0\,\bar{\rho} \to \mathbf{k} : (A, B) \mapsto \mathrm{Trace}(AB)$ is perfect. In particular, $\mathrm{Ad}^0\,\bar{\rho} \cong \mathrm{Ad}^0\,\bar{\rho}^*$ as representations. Thus $\mathrm{Ad}^0\,\bar{\rho}(1) \cong \mathrm{Ad}^0\,\bar{\rho}^*(1)$ as representations. By the Tate local duality this induces a perfect pairing

$$H^1(G_v, \mathrm{Ad}^0\,\bar{\rho}) \times H^1(G_v, \mathrm{Ad}^0\,\bar{\rho}(1)) \to H^2(G_v, \mathbf{k}(1)) \cong \mathbf{k}.$$

Definition 2. A *δ-lift of type* $(\mathcal{C}_v)_{v \in S_{\mathrm{f}}}$ is a δ-lift such that $\rho|_{G_v} \in \mathcal{C}_v$ for all $v \in S_{\mathrm{f}}$.

Definition 3. We define the *Selmer group* $H^1_{\{L_v\}}(G_{K,S}, \mathrm{Ad}^0\,\bar{\rho})$ to be the kernel of the map

$$H^1(G_{K,S}, \mathrm{Ad}^0\,\bar{\rho}) \to \bigoplus_{v \in S_{\mathrm{f}}} H^1(G_v, \mathrm{Ad}^0\,\bar{\rho})/L_v$$

and the *dual Selmer group* $H^1_{\{L_v^\perp\}}(G_{K,S}, \mathrm{Ad}^0\,\bar{\rho}(1))$ to be the kernel of the map

$$H^1(G_{K,S}, \mathrm{Ad}^0\,\bar{\rho}(1)) \to \bigoplus_{v \in S_{\mathrm{f}}} H^1(G_v, \mathrm{Ad}^0\,\bar{\rho}(1))/L_v^\perp$$

where $L_v^\perp \subset H^1(G_v, \mathrm{Ad}^0\,\bar{\rho}(1))$ is the annihilator of $L_v \subset H^1(G_v, \mathrm{Ad}^0\,\bar{\rho})$ under the above pairing.

Proposition 1. *Keep the above notation and assumptions. If*

$$H^1_{\{L_v^\perp\}}(G_{K,S}, \mathrm{Ad}^0\,\bar{\rho}(1)) = 0,$$

then there exists a δ-lift of $\bar{\rho}$ to $W(\mathbf{k})$ of type $(\mathcal{C}_v)_{v \in S_{\mathrm{f}}}$.

Proof. By Theorem 4.50 of [H] we have the exact sequence

$$H^1(G_{K,S}, \mathrm{Ad}^0 \bar{\rho}) \xrightarrow{\alpha} \bigoplus_{v \in S_{\mathrm{f}}} H^1(G_v, \mathrm{Ad}^0 \bar{\rho})/L_v \to H^1_{\{L_v^\perp\}}(G_{K,S}, \mathrm{Ad}^0 \bar{\rho}(1))^*$$
$$\to H^2(G_{K,S}, \mathrm{Ad}^0 \bar{\rho}) \xrightarrow{\beta} \bigoplus_{v \in S_{\mathrm{f}}} H^2(G_v, \mathrm{Ad}^0 \bar{\rho}).$$

Consequently, we see that the map α is surjective and the map β is injective. Now we construct δ-lifts ρ_n of $\bar{\rho}$ to $W(\mathbf{k})/p^n$ of type $(\mathcal{C}_v)_{v\in S_{\mathrm{f}}}$ inductively. By the condition (P1), there is nothing to prove for $n = 1$. Assume that there is a δ-lift ρ_{n-1} of $\bar{\rho}$ to $W(\mathbf{k})/p^{n-1}$ of type $(\mathcal{C}_v)_{v\in S_{\mathrm{f}}}$. By the condition (P6), for each $v \in S_{\mathrm{f}}$ we can lift $\rho_{n-1}|G_v$ to a continuous homomorphism $\rho_v : G_v \to \mathrm{GL}_2(W(\mathbf{k})/p^n)$ such that $(W(\mathbf{k})/p^n, \rho_v) \in \mathcal{C}_v$. Thus we can lift ρ_{n-1} to a continuous homomorphism $\rho : G_{K,S} \to \mathrm{GL}_2(W(\mathbf{k})/p^n)$ by injectivity of the map β. By surjectivity of the map α we may find a class $\phi \in H^1(G_{K,S}, \mathrm{Ad}^0 \bar{\rho})$ mapping to

$$([\rho_v - \rho|_{G_v}] \bmod L_v)_{v\in S_{\mathrm{f}}} \in \bigoplus_{v \in S_{\mathrm{f}}} H^1(G_v, \mathrm{Ad}^0 \bar{\rho})/L_v.$$

We define $\rho_n := (1 + \phi)\rho$. By the condition (P7) the representation ρ_n is a δ-lift of $\bar{\rho}$ to $W(\mathbf{k})/p^n$ of type $(\mathcal{C}_v)_{v\in S_{\mathrm{f}}}$. The induction is now complete. Then we have a δ-lift of $\bar{\rho}$ to $W(\mathbf{k})$ of type $(\mathcal{C}_v)_{v\in S_{\mathrm{f}}}$ by the condition (P5) and the proposition is proved. □

3 Local lifting problems

For a place v of K, consider a continuous homomorphism

$$\bar{\rho}_v : G_v \to \mathrm{GL}_2(\mathbf{k}).$$

We denote by $\widehat{\varepsilon} : G_v \to W(\mathbf{k})^\times$ the Teichmüller lift for any character $\varepsilon : G_v \to \mathbf{k}^\times$ and $\widehat{\mu} \in W(\mathbf{k})$ the Teichmüller lift for any element μ of $\mathbf{k}$. Let χ_p be the p-adic cyclotomic character.

In this section, for ramified places not dividing p and certain unramified places, we construct a good locally admissible pairs $(\mathcal{C}_v, L_v)$ with the $\delta_v := \widehat{\det \bar{\rho}_v}\widehat{\bar{\chi}_p}^{-1}\chi_p$, which will be used in Section 4. Let I_v be the inertia subgroup of G_v. We distinguish following three cases.

3.1 Case I

Suppose $\bar{\rho}_v$ is unramified and $v \nmid p$. Suppose that

$$\bar{\rho}_v(s) = \begin{pmatrix} \lambda & \lambda \\ 0 & \lambda \end{pmatrix}$$

and $q_v \equiv 1 \mod p$, where λ is an element of $\mathbf{k}^\times$ and s is a lift of the Frobenius automorphism in G_v/I_v and q_v is the order of the residue field of K_v. Note that any δ_v-lift of $\bar{\rho}_v$ factors through the Galois group $\mathrm{Gal}(K_v^{\mathrm{t}}/K_v)$ of the maximal tamely ramified extension K_v^{t} of K_v. Let P_v be the wild inertia subgroup of

G_v. Let t be a topological generator of I_v/P_v. The Galois group $\mathrm{Gal}(K_v^{\mathrm{t}}/K_v)$ is generated topologically by s and t with the relation $sts^{-1} = t^{q_v}$. We now define a homomorphism $\rho_v : G_v \to \mathrm{Gal}(K_v^{\mathrm{t}}/K_v) \to \mathrm{GL}_2(W(\mathbf{k})[[X]])$ by

$$s \mapsto \begin{pmatrix} \widehat{\lambda} q_v & \widehat{\lambda} \\ 0 & \widehat{\lambda} \end{pmatrix}$$

and

$$t \mapsto \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix}.$$

The images of s and t satisfy the relation $sts^{-1} = t^{q_v}$. We define a pair $(\mathcal{C}_v, L_v)$. The functor $\mathcal{C}_v : \mathcal{A} \to \mathbf{Sets}$ is given by

$$\mathcal{C}_v(R) := \{\rho : G_v \to \mathrm{GL}_2(R) \mid \text{there are } \alpha \in \mathrm{Hom}_{\mathcal{A}}(W(\mathbf{k})[[X]], R) \text{ and } M \in 1 + \mathrm{M}_2(\mathfrak{m}_R) \text{ such that } \rho = M(\alpha \circ \rho_v)M^{-1}\}.$$

Moreover, if $\rho_0 : G_v \to \mathrm{GL}_2(\mathbf{k}[X]/(X^2))$ denotes the trivial lift of $\bar{\rho}_v$, we define a subspace $L_v \subset H^1(G_v, \mathrm{Ad}^0\,\bar{\rho}_v)$ to be the set

$$\{[c] \in H^1(G_v, \mathrm{Ad}^0\,\bar{\rho}_v) \mid (1 + Xc)\rho_0 \in \mathcal{C}_v(\mathbf{k}[X]/(X^2))\}.$$

Lemma 1. *We have*
(i) $\dim_{\mathbf{k}} L_v = \dim_{\mathbf{k}} H^1(G_v/I_v, \mathrm{Ad}^0\,\bar{\rho}_v) = 1$.
(ii) *The pair* $(\mathcal{C}_v, L_v)$ *satisfies the conditions* (P1)-(P7) *of Definition 1.*

Proof. (i) First we prove that $\dim_{\mathbf{k}} H^1(G_v/I_v, \mathrm{Ad}^0\,\bar{\rho}_v) = 1$. By Proposition 18 of [S2] the dimension of $H^1(G_v/I_v, \mathrm{Ad}^0\,\bar{\rho}_v)$ is the same as that of $H^0(G_v, \mathrm{Ad}^0\,\bar{\rho}_v)$. Thus it suffices to show that $H^0(G_v, \mathrm{Ad}^0\,\bar{\rho}_v)$ is one-dimensional. This follows from

$$\begin{pmatrix} \lambda & \lambda \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} 1/\lambda & -1/\lambda \\ 0 & 1/\lambda \end{pmatrix} = \begin{pmatrix} a + c & -2a + b - c \\ c & -(a + c) \end{pmatrix},$$

where $a, b, c \in \mathbf{k}$.

Next we prove that $\dim_{\mathbf{k}} L_v = 1$. Let $f_1 : W[[X]] \to \mathbf{k}[X]/(X^2)$ be the morphism in $\mathcal{A}$ determined by $f_1(X) = X$. We define $\rho_1 : G_v \to \mathrm{GL}_2(\mathbf{k}[X]/(X^2))$ by the composition $f_1 \circ \rho_v$. The images of s and t satisfy the relation $sts^{-1} = t^{q_v}$. Let c_1 be the 1-cocycle corresponding to ρ_1. The space L_v is spanned by the class of c_1. Thus we have $\dim_{\mathbf{k}} L_v = 1$.

(ii) The conditions (P1), (P2), (P3), (P6) and (P7) follow from the definition of $(\mathcal{C}_v, L_v)$.

First we prove the condition (P4). Suppose that we have rings $(R_1, \mathfrak{m}_{R_1}), (R_2, \mathfrak{m}_{R_2}) \in \mathcal{A}$, lifts $\rho_i \in \mathcal{C}_v(R_i)$, ideals $I_i \subset R_i$, and an identification $\phi : R_1/I_1 \xrightarrow{\sim} R_2/I_2$ under which $\rho_1 \pmod{I_1} = \rho_2 \pmod{I_2}$. Take $\alpha_i \in \mathrm{Hom}_{\mathcal{A}}(W(\mathbf{k})[[X]], R_i)$ and $M_i \in 1 + \mathrm{M}_2(\mathfrak{m}_{R_i})$ such that $\rho_i = M_i(\alpha_i \circ \rho_v)M_i^{-1}$, $i = 1, 2$. We claim that there exist $\alpha \in \mathrm{Hom}_{\mathcal{A}}(W(\mathbf{k})[[X]], R_3)$ and $M \in 1 + \mathrm{M}_2(\mathfrak{m}_{R_3})$ such that $M(\alpha \circ \rho_v)M^{-1} = \rho_1 \oplus \rho_2$. By conjugating ρ_1 by some lift of $M_2 \pmod{I_2}$ to R_1, we may assume that $M_2 = 1$. Since $\alpha_1 \circ \rho_v(s) = \alpha_2 \circ \rho_v(s)$, the matrix $M_1 \pmod{I_1}$ commutes with $(\alpha_1 \pmod{I_1}) \circ \rho_v(s)$. Let $\begin{pmatrix} 1 + m_1 & m_2 \\ 0 & 1 + m_3 \end{pmatrix} \in 1 + \mathrm{M}_2(\mathfrak{m}_{R_1})$ be a lift of $M_1 \pmod{I_1}$. Put $M_1' := \begin{pmatrix} 1 + m_1 & m_2 \\ 0 & 1 + m_3 - x \end{pmatrix}$, where $x := (q_v - 1)m_2 - m_1 + m_3$. Note that $x \in I_1$. Then $M_1' \in 1 + \mathrm{M}_2(\mathfrak{m}_{R_1})$ commutes with $\alpha_1 \circ \rho_v(s)$. We now replace M_1 by $\widetilde{M_1} := M_1 M_1'^{-1}$ and α_1 by some $\widetilde{\alpha}_1 : W(\mathbf{k})[[X]] \to R_1$ such that $\widetilde{M}_1(\widetilde{\alpha}_1 \circ \rho_v)\widetilde{M}_1^{-1} = M_1(\alpha_1 \circ \rho_v)M_1^{-1}$. Defining $M := (\widetilde{M}_1, 1) \in 1 + \mathrm{M}_2(\mathfrak{m}_{R_3})$ and $\alpha := (\widetilde{\alpha}_1, \alpha_2) : W(\mathbf{k})[[X]] \to R_3$, the condition (P4) is verified.

Next we prove the condition (P5). Suppose that we have a ring $R \in \mathcal{A}$ and a δ_v-lift ρ of $\bar{\rho}_v$ to R such that each $\rho \pmod{\mathfrak{m}_R^n} \in \mathcal{C}_v(R/\mathfrak{m}_R^n)$. Put $\rho_n := \rho \pmod{\mathfrak{m}_R^n}$. Take $\alpha_n \in \mathrm{Hom}_{\mathcal{A}}(W(\mathbf{k})[[X]], R/\mathfrak{m}_R^n)$ and $M_n \in 1 + \mathrm{M}_2(\mathfrak{m}_R/\mathfrak{m}_R^n)$ such that $\rho_n = M_n(\alpha_n \circ \rho_v)M_n^{-1}$. We claim that there exist $\alpha \in \mathrm{Hom}_{\mathcal{A}}(R_v, R)$ and $M \in 1 + \mathrm{M}_2(\mathfrak{m}_R)$ such that $M(\alpha \circ \rho_v)M^{-1} = \rho$. Put $S_n := \{(\alpha_n', M_n') \mid \rho_n = M_n'(\alpha_n' \circ \rho_v)M_n'^{-1}\}$. Since $\mathcal{C}_v(R/\mathfrak{m}_R^n)$ is finite, S_n is finite. For each n, S_n is not empty set. Thus $\varprojlim_n S_n$ is not empty set, the condition (P5) is verified. □

3.2 Case II

Suppose $\bar{\rho}_v$ is ramified and $v \nmid p$. In addition, suppose $\bar{\rho}_v(I_v)$ is of order prime to p. Define the functor $\mathcal{C}_v : \mathcal{A} \to \mathbf{Sets}$ by

$$\mathcal{C}_v(R) := \{\rho : G_v \to \mathrm{GL}_2(R) \mid \rho \pmod{\mathfrak{m}_R} = \bar{\rho}_v, \rho(I_v) \xrightarrow{\sim} \bar{\rho}_v(I_v), \det\rho = \delta_v\}.$$

Moreover, if $\rho_0 : G_v \to \mathrm{GL}_2(\mathbf{k}[X]/(X^2))$ denotes the trivial lift of $\bar{\rho}_v$, we define a subspace $L_v \subset H^1(G_v, \mathrm{Ad}^0\,\bar{\rho}_v)$ to be the set

$$\{[c] \in H^1(G_v, \mathrm{Ad}^0\,\bar{\rho}_v) \mid (1 + Xc)\rho_0 \in \mathcal{C}_v(\mathbf{k}[X]/(X^2))\}.$$

Lemma 2. *We have*
(i) $\dim_{\mathbf{k}} L_v = \dim_{\mathbf{k}} H^0(G_v, \mathrm{Ad}^0\,\bar{\rho}_v)$.
(ii) *The pair* $(\mathcal{C}_v, L_v)$ *satisfies the conditions* (P1)-(P7) *of Definition 1.*

Proof. This lemma follows from the definitions and the Schur-Zassenhaus theorem. □

3.3 Case III

Suppose $\bar{\rho}_v$ is ramified and $v \nmid p$. In addition, suppose the order of $\bar{\rho}_v(I_v)$ is divisible by p. By Lemma 3.1 of [G], since $p \geq 7$, we may assume that $\bar{\rho}_v$ is given by the form

$$\bar{\rho}_v = \begin{pmatrix} \varphi\bar{\chi}_p & \gamma \\ 0 & \varphi \end{pmatrix},$$

for a character $\varphi : G_v \to \mathbf{k}^\times$ and a nonzero continuous function $\gamma : G_v \to \mathbf{k}$. The functor $\mathcal{C}_v : \mathcal{A} \to \mathbf{Sets}$ is given by

$$\begin{aligned} \mathcal{C}_v(R) := \{\rho : G_v \to \mathrm{GL}_2(R) \mid &\text{ there are } \widetilde{\gamma} \in \mathrm{Map}(G_v, R) \text{ and } M \in 1 + \mathrm{M}_2(\mathfrak{m}_R) \\ &\text{such that } \rho = M \begin{pmatrix} \widehat{\varphi}\chi_p & \widetilde{\gamma} \\ 0 & \widehat{\varphi} \end{pmatrix} M^{-1}, \widetilde{\gamma} \bmod \mathfrak{m}_R = \gamma\}. \end{aligned}$$

Moreover, if $\rho_0 : G_v \to \mathrm{GL}_2(\mathbf{k}[X]/(X^2))$ denotes the trivial lift of $\bar{\rho}_v$, we define a subspace $L_v \subset H^1(G_v, \mathrm{Ad}^0\,\bar{\rho}_v)$ to be the set

$$\{[c] \in H^1(G_v, \mathrm{Ad}^0\,\bar{\rho}_v) \mid (1 + Xc)\rho_0 \in \mathcal{C}_v(\mathbf{k}[X]/(X^2))\}.$$

Lemma 3. *We have*
(i) $\dim_{\mathbf{k}} L_v = \dim_{\mathbf{k}} H^0(G_v, \mathrm{Ad}^0\,\bar{\rho}_v)$.
(ii) *The pair* $(\mathcal{C}_v, L_v)$ *satisfies the conditions* (P1)-(P7) *of Definition 1.*

Proof. The proof of this lemma is almost identical argument as in [T, Section 1(E3)]. $\square$

4 Lifting theorem over arbitrary number fields

In this section, we give a generalization of Theorem 1 of [R1] to arbitrary number fields.

We define $\delta : G_{K,S} \to W(\mathbf{k})^\times$ by $\widehat{\det\bar{\rho}}\widehat{\chi}_p^{-1}\chi_p$. Throughout this section, we consider lifts of a fixed determinant δ and we always assume the following:

- The order of the image of $\bar{\rho}$ is divisible by p.

By the Schur-Zassenhaus theorem, if the order of the image of $\bar{\rho}$ is prime to p, we can find a lift to $W(\mathbf{k})$ of $\bar{\rho}$. Since $p \geq 7$ and the order of the image of $\bar{\rho}$ is divisible by p, we see from Section 260 of [D] that the image of $\bar{\rho}$ is contained in the Borel subgroup of $\mathrm{GL}_2(\mathbf{k})$ or the projective image of $\bar{\rho}$ is conjugate to either $\mathrm{PGL}_2(\mathbb{F}_{p^r})$ or $\mathrm{PSL}_2(\mathbb{F}_{p^r})$ for some $r \in \mathbb{Z}_{>0}$. In the Borel case, by Theorem 2 of [K] we have a lift of $\bar{\rho}$ to $W(\mathbf{k})$. Thus we may assume that the projective image of $\bar{\rho}$ is equal to $\mathrm{PSL}_2(\mathbb{F}_{p^r})$ or $\mathrm{PGL}_2(\mathbb{F}_{p^r})$. Then, by Lemma 17 of [R1], $\mathrm{Ad}^0\,\bar{\rho}$ is an irreducible $G_{K,S}$-module. (Note that one may replace the assumption that the image of $\bar{\rho}$ contains $\mathrm{SL}_2(\mathbf{k})$ in [R1] with the assumption that the projective image of $\bar{\rho}$ contains $\mathrm{PSL}_2(\mathbb{F}_p)$ without affecting the proof.) The irreducibility of $\mathrm{Ad}^0\,\bar{\rho}$ implies that of $\mathrm{Ad}^0\,\bar{\rho}(1)$.

Let $K(\mathrm{Ad}^0\,\bar{\rho})$ be the fixed field of $\mathrm{Ker}(\mathrm{Ad}^0\,\bar{\rho})$. Put $E = K(\mathrm{Ad}^0\,\bar{\rho})K(\mu_p)$ and $D = K(\mathrm{Ad}^0\,\bar{\rho}) \cap K(\mu_p)$.

Lemma 4. *We have*

$$H^1(\mathrm{Gal}(E/K), \mathrm{Ad}^0\,\bar{\rho}) = H^1(\mathrm{Gal}(E/K), \mathrm{Ad}^0\,\bar{\rho}(1)) = 0.$$

Proof. First we prove that $H^1(\mathrm{Gal}(E/K), \mathrm{Ad}^0\,\bar{\rho}) = 0$. It suffices to show that $H^1(\mathrm{SL}_2(\mathbb{F}_{p^r}), \mathrm{Ad}^0\,\bar{\rho}) = 0$ and $H^1(\mathrm{GL}_2(\mathbb{F}_{p^r}), \mathrm{Ad}^0\,\bar{\rho}) = 0$, where $\mathrm{GL}_2(\mathbb{F}_{p^r})$ and $\mathrm{SL}_2(\mathbb{F}_{p^r})$ act on $\mathrm{Ad}^0\,\bar{\rho}$ by conjugation. By Lemma 2.48 of [DDT], we see $H^1(\mathrm{SL}_2(\mathbb{F}_{p^r}), \mathrm{Ad}^0\,\bar{\rho}) = 0$. Since the index of $\mathrm{SL}_2(\mathbb{F}_{p^r})$ in $\mathrm{GL}_2(\mathbb{F}_{p^r})$ is prime to p, we have $H^1(\mathrm{GL}_2(\mathbb{F}_{p^r}), \mathrm{Ad}^0\,\bar{\rho}) = 0$.

Next we prove that $H^1(\mathrm{Gal}(E/K), \mathrm{Ad}^0\,\bar{\rho}(1)) = 0$. As $D \subset K(\mu_p)$, we see $\mathrm{Gal}(K(\mathrm{Ad}^0\,\bar{\rho})/D)$ contains the commutator subgroup of $\mathrm{Gal}(K(\mathrm{Ad}^0\,\bar{\rho})/K)$. Since the projective image of $\bar{\rho}$ is equal to $\mathrm{PSL}_2(\mathbb{F}_{p^r})$ or $\mathrm{PGL}_2(\mathbb{F}_{p^r})$, we see this commutator subgroup is just $\mathrm{PSL}_2(\mathbb{F}_{p^r})$. Thus $\mathrm{Gal}(K(\mathrm{Ad}^0\,\bar{\rho})/K)/\mathrm{PSL}_2(\mathbb{F}_{p^r}) \to \mathrm{Gal}(D/K)$ is surjective, and so $[D : K] = 1$ or 2. Assume that $[K(\mu_p) : K] = 1$, then $H^1(\mathrm{Gal}(E/K), \mathrm{Ad}^0\,\bar{\rho}(1))$ is isomorphic to $H^1(\mathrm{Gal}(E/K), \mathrm{Ad}^0\,\bar{\rho})$. Consequently $H^1(\mathrm{Gal}(E/K), \mathrm{Ad}^0\,\bar{\rho}(1)) = 0$.

Assume that $[K(\mu_p) : K] \geq 3$, or $[K(\mu_p) : K] = 2$ and $[D : K] = 1$. We apply the inflation-restriction sequence to $\mathrm{Gal}(E/K)$ and its normal subgroup $\mathrm{Gal}(E/K(\mathrm{Ad}^0\,\bar{\rho}))$. Since $\mathrm{Gal}(K_S/E)$ fixes $\mathrm{Ad}^0\,\bar{\rho}(1)$ we see $\mathrm{Ad}^0\,\bar{\rho}(1)^{\mathrm{Gal}(E/K(\mathrm{Ad}^0\,\bar{\rho}))} = \mathrm{Ad}^0\,\bar{\rho}(1)^{\mathrm{Gal}(K_S/K(\mathrm{Ad}^0\,\bar{\rho}))}$. We get the exact sequence

$$0 \to H^1(\mathrm{Gal}(K(\mathrm{Ad}^0\,\bar{\rho})/K), \mathrm{Ad}^0\,\bar{\rho}(1)^{\mathrm{Gal}(K_S/K(\mathrm{Ad}^0\,\bar{\rho}))}) \to H^1(\mathrm{Gal}(E/K), \mathrm{Ad}^0\,\bar{\rho}(1))$$
$$\to H^1(\mathrm{Gal}(E/K(\mathrm{Ad}^0\,\bar{\rho})), \mathrm{Ad}^0\,\bar{\rho}(1))^{\mathrm{Gal}(K(\mathrm{Ad}^0\,\bar{\rho})/K)}.$$

The last term is trivial as $\mathrm{Gal}(E/K(\mathrm{Ad}^0\bar{\rho}))$ has order prime to p. As $\mathrm{Gal}(K_S/K(\mathrm{Ad}^0\bar{\rho}))$ acts trivially on $\mathrm{Ad}^0\bar{\rho}$ we see the action of $\mathrm{Gal}(K_S/K(\mathrm{Ad}^0\bar{\rho}))$ is $\chi_p|_{\mathrm{Gal}(K_S/K(\mathrm{Ad}^0\bar{\rho}))}$, which is nontrivial, so $\mathrm{Ad}^0\bar{\rho}(1)^{\mathrm{Gal}(K_S/K(\mathrm{Ad}^0\bar{\rho}))} = 0$. Thus the left term in the sequence is trivial, so $H^1(\mathrm{Gal}(E/K), \mathrm{Ad}^0\bar{\rho}(1)) = 0$.

Assume that $[K(\mu_p) : K] = 2$ and $[D : K] = 2$, then we have $K(\mu_p) = D$. Note that $\mathrm{PSL}_2(\mathbb{F}_{p^r})$ has no non-trivial abelian quotients. If the projective image of $\bar{\rho}$ is $\mathrm{PSL}_2(\mathbb{F}_{p^r})$ for some $r \in \mathbb{Z}_{>0}$, then $\mathrm{Gal}(E/K)$ has no non-trivial abelian quotients. This contradicts the assumption that $[K(\mu_p) : K] = 2$. Hence, we assume that the projective image of $\bar{\rho}$ is $\mathrm{PGL}_2(\mathbb{F}_{p^r})$ for some $r \in \mathbb{Z}_{>0}$. Since the index of $\mathrm{PSL}_2(\mathbb{F}_{p^r})$ in $\mathrm{PGL}_2(\mathbb{F}_{p^r})$ is equal to the index of $\mathrm{Gal}(E/K(\mu_p))$ in $\mathrm{Gal}(E/K)$, $\mathrm{Gal}(E/K(\mu_p))$ is isomorphic to $\mathrm{PSL}_2(\mathbb{F}_{p^r})$. We have

$$H^1(\mathrm{Gal}(E/K), \mathrm{Ad}^0\bar{\rho}(1)) \hookrightarrow H^1(\mathrm{Gal}(E/K(\mu_p)), \mathrm{Ad}^0\bar{\rho}(1)).$$

Since $\mathrm{Ad}^0\bar{\rho}(1)$ is isomorphic to $\mathrm{Ad}^0\bar{\rho}$ as a $\mathrm{Gal}(E/K(\mu_p))$-module and the cohomology group $H^1(\mathrm{Gal}(E/K(\mu_p)), \mathrm{Ad}^0\bar{\rho})$ is zero, the proof is complete. □

Lemma 5. *If a pair $(\mathcal{C}_v, L_v)$ which is locally admissible is given for each $v \in S_{\mathrm{f}}$ and each elements $\phi \in H^1_{\{L_v^\perp\}}(G_{K,S}, \mathrm{Ad}^0\bar{\rho}(1))$ and $\psi \in H^1_{\{L_v\}}(G_{K,S}, \mathrm{Ad}^0\bar{\rho})$ are not zero, then we can find a prime $w \notin S$ and a locally admissible pair $(\mathcal{C}_w, L_w)$ such that*
(1) $\dim_{\mathbf{k}} H^1(G_w/I_w, \mathrm{Ad}^0\bar{\rho}) = \dim_{\mathbf{k}} L_w = 1$,
(2) *the image of ψ in $H^1(G_w/I_w, \mathrm{Ad}^0\bar{\rho})$ is not zero,*
(3) *the image of ϕ in $H^1(G_w, \mathrm{Ad}^0\bar{\rho}(1))/L_w^\perp$ is not zero.*

Proof. Note that Lemma 4 implies that the restrictions of the cocycles ψ and ϕ are non-zero homomorphisms $\phi : \mathrm{Gal}(K_S/E) \to \mathrm{Ad}^0\bar{\rho}(1)$ and $\psi : \mathrm{Gal}(K_S/E) \to \mathrm{Ad}^0\bar{\rho}$. Let E_ϕ and E_ψ be the fixed fields of the respective kernels. Then, $\mathrm{Gal}(E_\phi/E) \to \mathrm{Ad}^0\bar{\rho}(1)$ and $\mathrm{Gal}(E_\psi/E) \to \mathrm{Ad}^0\bar{\rho}$ are injective homomorphisms of $\mathbb{F}_p[G_{K,S}]$-modules. Since $\mathrm{Ad}^0\bar{\rho}$ is irreducible $G_{K,S}$-module, these morphisms are bijective, and we see $E_\phi \cap E_\psi = E_\psi (= E_\phi)$ or E. If the intersection is E, then $\mathrm{Gal}(E_\phi E_\psi/E)$ is isomorphic to $\mathrm{Gal}(E_\phi/E) \times \mathrm{Gal}(E_\psi/E)$. If the intersection is E_ψ, then $\mathrm{Gal}(E_\phi E_\psi/E)$ is isomorphic to $\mathrm{Gal}(E_\psi/E)$ and $\mathrm{Gal}(E_\phi/E)$. Therefore, $\mathrm{Gal}(E_\phi E_\psi/E)$ may be regarded as a $\mathbf{k}[\mathrm{Gal}(E/K)]$-module, moreover, natural homomorphisms $\mathrm{Gal}(E_\phi E_\psi/E) \to \mathrm{Ad}^0\bar{\rho}(1)$ and $\mathrm{Gal}(E_\phi E_\psi/E) \to \mathrm{Ad}^0\bar{\rho}$ are surjective. Since $\mathrm{PSL}_2(\mathbb{F}_{p^r})$ has no non-trivial abelian quotients, the image of the morphism $\widetilde{\bar{\rho}} \times \chi_p : G_{K,S} \to \mathrm{PGL}_2(\mathbf{k}) \times \mathbf{k}^\times$ contains $\mathrm{PSL}_2(\mathbb{F}_{p^r}) \times 1$, where $\widetilde{\bar{\rho}}$ is the projective image of $\bar{\rho}$ and χ_p is the mod p cyclotomic character of $G_{K,S}$. Thus there is an element $\sigma \in \mathrm{Gal}(E/K)$ such that $\chi_p(\sigma) = 1$ and $\bar{\rho}(\sigma) = \begin{pmatrix} \lambda & \lambda \\ 0 & \lambda \end{pmatrix}$, for some element $\lambda \in \mathbf{k}^\times$. We denote by $\widetilde{\sigma}$ a lift to $\mathrm{Gal}(E_\phi E_\psi/K)$ of σ. Let L be the subset of $\mathrm{Ad}^0\bar{\rho}$ whose elements have the form $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ and let L' be the subset of $\mathrm{Ad}^0\bar{\rho}(1)$ whose elements have the form $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$. Since L and L' are two-dimensional, there exists $\tau \in \mathrm{Gal}(E_\phi E_\psi/E)$ such that $\psi(\tau) \notin -\psi(\widetilde{\sigma}) + L$ and $\phi(\tau) \notin -\phi(\widetilde{\sigma}) + L'$.

By the Čebotarev density theorem, we can choose a place $w \notin S$ which is unramified in $E_\phi E_\psi/K$ such that $\mathrm{Frob}_w = \tau\widetilde{\sigma}$. Take $\mathcal{C}_w$ and L_w as in Case I. By Lemma 1 of this paper and Lemma 4.8 of [BK], it follows that $(w, \mathcal{C}_w, L_w)$

has the desired properties. (Note that one may replace function fields in [BK] with number fields without affecting the proof.) □

Lemma 6. *Suppose that one is given locally admissible pairs* $(\mathcal{C}_v, L_v)_{v\in S_{\mathrm{f}}}$ *such that*

$$\sum_{v\in S_{\mathrm{f}}} \dim_{\mathbf{k}} L_v \geq \sum_{v\in S} \dim_{\mathbf{k}} H^0(G_v, \mathrm{Ad}^0\,\bar{\rho}).$$

Then we can find a finite set of places $T \supset S$ *and locally admissible pairs* $(\mathcal{C}_v, L_v)_{v\in T\smallsetminus S}$ *such that*

$$H^1_{\{L_v^\perp\}}(G_{K,T}, \mathrm{Ad}^0\,\bar{\rho}(1)) = 0.$$

Proof. Suppose that $0 \neq \phi \in H^1_{\{L_v^\perp\}}(G_{K,S}, \mathrm{Ad}^0\,\bar{\rho}(1))$. By the assumption of the lemma and Theorem 4.50 of [H], we see that $\dim_{\mathbf{k}} H^1_{\{L_v\}}(G_{K,S}, \mathrm{Ad}^0\,\bar{\rho}) \geq \dim_{\mathbf{k}} H^1_{\{L_v^\perp\}}(G_{K,S}, \mathrm{Ad}^0\,\bar{\rho}(1))$. Then we can find $0 \neq \psi \in H^1_{\{L_v\}}(G_{K,S}, \mathrm{Ad}^0\,\bar{\rho})$. Thus we can find a place $w \notin S$ and a locally admissible pair $(\mathcal{C}_w, L_w)$ such that
(1) $\dim_{\mathbf{k}} H^1(G_w/I_w, \mathrm{Ad}^0\,\bar{\rho}) = \dim_{\mathbf{k}} L_w$,
(2) $H^1_{\{L_v\}}(G_{K,S}, \mathrm{Ad}^0\,\bar{\rho}) \to H^1(G_w/I_w, \mathrm{Ad}^0\,\bar{\rho})$ is surjective,
(3) the image of ϕ in $H^1(G_w, \mathrm{Ad}^0\,\bar{\rho}(1))/L_w^\perp$ is not zero,
by Lemma 5. We have an injection

$$H^1_{\{L_v^\perp\}}(G_{K,S}, \mathrm{Ad}^0\,\bar{\rho}(1)) \hookrightarrow H^1_{\{L_v^\perp\}\cup\{H^1(G_w,\mathrm{Ad}^0\,\bar{\rho}(1))\}}(G_{K,S\cup\{w\}}, \mathrm{Ad}^0\,\bar{\rho}(1))$$

and we see that its cokernel has order equal to

$$\#\mathrm{Coker}(H^1_{\{L_v\}}(G_{K,S}, \mathrm{Ad}^0\,\bar{\rho}) \to H^1(G_w/I_w, \mathrm{Ad}^0\,\bar{\rho})),$$

by applying Theorem 4.50 of [H] to

$$H^1_{\{L_v^\perp\}}(G_{K,S}, \mathrm{Ad}^0\,\bar{\rho}(1))$$

and

$$H^1_{\{L_v^\perp\}\cup\{H^1(G_w,\mathrm{Ad}^0\,\bar{\rho}(1))\}}(G_{K,S\cup\{w\}}, \mathrm{Ad}^0\,\bar{\rho}(1)).$$

Thus

$$H^1_{\{L_v^\perp\}}(G_{K,S}, \mathrm{Ad}^0\,\bar{\rho}(1)) = H^1_{\{L_v^\perp\}\cup\{H^1(G_w,\mathrm{Ad}^0\,\bar{\rho}(1))\}}(G_{K,S\cup\{w\}}, \mathrm{Ad}^0\,\bar{\rho}(1)),$$

and we obtain an exact sequence

$$\begin{aligned}0 &\to H^1_{\{L_v^\perp\}\cup\{L_w^\perp\}}(G_{K,S\cup\{w\}}, \mathrm{Ad}^0\,\bar{\rho}(1)) \to H^1_{\{L_v^\perp\}}(G_{K,S}, \mathrm{Ad}^0\,\bar{\rho}(1)) \\ &\to H^1(G_w, \mathrm{Ad}^0\,\bar{\rho}(1))/L_w^\perp.\end{aligned}$$

Hence $\phi \notin H^1_{\{L_v^\perp\}\cup\{L_w^\perp\}}(G_{K,S\cup\{w\}}, \mathrm{Ad}^0\,\bar{\rho}(1)) \subset H^1_{\{L_v^\perp\}}(G_{K,S}, \mathrm{Ad}^0\,\bar{\rho}(1))$. The lemma will follow by repeating such a computation. □

Let S' denote the set of places of K consisting of the places above p, the infinite places and the places at which $\bar{\rho}$ is ramified.

Proof of Theorem. This follows almost at once from Proposition 1 and Lemma 6. For each places v satisfying $v \in S'_{\mathrm{f}}$ and $v \nmid p$, take $\mathcal{C}_v$ and L_v as in Case II or Case III. For places $v \mid p$, take $\mathcal{C}_v$ and L_v as the collection of all $\delta|_{G_v}$-lifts of $\bar{\rho}|_{G_v}$ and $H^1(G_v, \mathrm{Ad}^0\,\bar{\rho})$, respectively. By Theorem 4.52 of [H] and the assumption of Theorem, we have

$$\sum_{v|p} \dim_{\mathbf{k}} L_v = \sum_{v|p} \dim_{\mathbf{k}} H^0(G_v, \mathrm{Ad}^0\,\bar{\rho}) + \sum_{v|p} [K_v : \mathbb{Q}_p] \dim_{\mathbf{k}} \mathrm{Ad}^0\,\bar{\rho}$$

and thus we obtain

$$\sum_{v \in S'_{\mathrm{f}}} \dim_{\mathbf{k}} L_v \geq \sum_{v \in S'} \dim_{\mathbf{k}} H^0(G_v, \mathrm{Ad}^0\,\bar{\rho}).$$

□

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