On $v$-adic periods of $t$-motives

Yoshinori Mishiba*

May 31, 2011

Abstract

In this paper, we prove the equality between the transcendental degree of the field generated by the $v$-adic periods of a $t$-motive $M$ and the dimension of the Tannakian Galois group for $M$, where $v$ is a “finite” place of the rational function field over a finite field. As an application, we prove the algebraic independence of certain “formal” polylogarithms.

Contents

1 Introduction 2

2 Notations and terminology 3
   2.1 Table of symbols . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 3
   2.2 Action . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 4
   2.3 Base change . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 4

3 $\varphi$-modules 4
   3.1 étale $\varphi$-modules . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 4
   3.2 $L$-triviality . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 6
   3.3 $v$-adic case . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 10

4 Frobenius equations 12
   4.1 The group $\Gamma$ . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 12
   4.2 $\Gamma$-action . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 19

5 The group $\Gamma$ and $\varphi$-modules 21
   5.1 General case . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 21
   5.2 $v$-adic case . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 27

6 $v$-adic criterion 28

7 Algebraic independence of formal polylogarithms 32

---

*Graduate School of Mathematics, Kyushu University, 744, Motooka, Nishi-ku, Fukuoka, 819-0395, JAPAN
e-mail: y-mishiba@math.kyushu-u.ac.jp
1 Introduction

Let \( F_q \) be the finite field with \( q \) elements, \( \theta \) and \( t \) be variables independent from each other, and \( v \in F_q[t] \) a fixed monic irreducible polynomial of degree \( d \). Let \( M \) be a rigid analytically trivial \( t \)-motive over \( \overline{F_q(\theta)} \). Then there exists an \( \infty \)-adic period matrix for the Betti realization of \( M \). Set \( \Lambda \) to be the field generated by the components of this matrix over \( \overline{F_q(\theta)(t)} \). Set \( \Gamma \) to be the Tannakian Galois group of \( M \) with respect to the Betti realization. Papanikolas [10] shows that the transcendental degree of \( \Lambda \) over \( \overline{F_q(\theta)(t)} \) coincides with the dimension of \( \Gamma \). In this paper, we prove the \( \nu \)-adic analogue of this theorem.

Let \( K/F_q \) be a regular extension of fields. We set \( K^{\text{sep}}[t]_v := \varprojlim (K^{\text{sep}}[t]/v^n) \) and \( K_v^{\text{sep}} := F_q(t) \otimes_{F_q[t]} K^{\text{sep}}[t]_v \), where \( K^{\text{sep}} \) is a separable closure of \( K \). We also define \( F_q(t)_v \) and \( K(t)_v \) by the same way. Let \( \sigma \) be the ring endomorphism \( \sum a_i t^i \mapsto \sum a^\sigma_i t^i \) of \( K^{\text{sep}}[t]_v \). Then \( \sigma \) naturally extends to an endomorphism of \( K_v^{\text{sep}} \), also denoted by \( \sigma \).

A \( \varphi \)-module over \( K(t)_v \) is a pair \( (\varphi, \varphi) \) (or simply \( \varphi \)) where \( M \) is a \( K(t)_v \)-vector space and \( \varphi : M \to M \) is an additive map such that \( \varphi(ax) = \varphi(a) \varphi(x) \) for all \( a \in K(t)_v \) and \( x \in M \). A morphism of \( \varphi \)-modules is a \( K(t)_v \)-linear map which is compatible with the \( \varphi \)'s. A tensor product of two \( \varphi \)-modules is defined naturally.

For any \( \varphi \)-module \( M \), we define the \( \varphi \)-adic realization of \( M \):

\[
V(M) := (K_v^{\text{sep}}(t)_v \otimes_{K(t)_v} M)^{\varphi},
\]

where \( \varphi \) acts on \( K_v^{\text{sep}}(t)_v \otimes_{K(t)_v} M \) by \( \sigma \otimes \varphi \) and \((-)^\varphi \) is the \( \varphi \)-fixed part. Then there exists a natural map

\[
i_M : K_v^{\text{sep}}(t)_v \otimes_{F_q(t)_v} V(M) \to K_v^{\text{sep}}(t)_v \otimes_{K(t)_v} M.
\]

We can prove that \( i_M \) is injective for each \( \varphi \)-module \( M \). A \( \varphi \)-module \( M \) is said to be \( K_v^{\text{sep}}(t)_v \)-trivial if \( M \) is finite-dimensional over \( K(t)_v \) and \( i_M \) is an isomorphism. Then the category of \( K_v^{\text{sep}}(t)_v \)-trivial \( \varphi \)-modules over \( K(t)_v \), equipped with the functor \( V \) forms a neutral Tannakian category over \( F_q(t)_v \). For any \( K_v^{\text{sep}}(t)_v \)-trivial \( \varphi \)-module \( M \), we denote by \( \Gamma_M \) the Tannakian Galois group of the Tannakian subcategory of \( K_v^{\text{sep}}(t)_v \)-trivial \( \varphi \)-modules generated by \( M \) (see Subsections 3.2 and 3.3).

Let \( M \) be a finite-dimensional \( \varphi \)-module and \( m \in \text{Mat}_{r \times 1}(M) \) a \( K(t)_v \)-basis of \( M \). Then there exists a matrix \( \Phi \in \text{Mat}_{r \times \infty}(K(t)_v) \) such that \( \varphi m = \Phi m \). If \( M \) is \( K_v^{\text{sep}}(t)_v \)-trivial, we can take a matrix \( \Psi = (\Psi_{ij})_{i,j} \in \text{GL}_r(K_v^{\text{sep}}(t)_v) \) such that \( \Psi^{-1} m \) forms an \( F_q(t)_v \)-basis of \( V(M) \). The entries of this matrix are called \( \varphi \)-adic periods of \( M \), which are our main objects of study in this paper. We set

\[
\Sigma := K(t)_v[\Psi, 1/\det \Psi] := K(t)_v[\Psi_{11}, \Psi_{12}, \ldots, \Psi_{rr}, 1/\det \Psi] \subset K_v^{\text{sep}}(t)_v.
\]

Then \( \Sigma \) is stable under the \( \sigma \)-action. For any \( F_q(t)_v \)-algebra \( R \) and \( S \), we set \( S^{(R)} := R \otimes_{F_q(t)_v} S \). If \( \sigma \) acts on \( S \), we define the \( \sigma \)-action on \( S^{(R)} \) by \( \id \otimes \sigma \). Set \( \Gamma(R) := \text{Aut}_\sigma(\Sigma^{(R)}/K(t)_v^{(R)}) \) the group of automorphisms of \( \Sigma^{(R)} \) over \( K(t)_v^{(R)} \) that commute with \( \sigma \). Then \( \Gamma \) forms a functor from the category of \( F_q(t)_v \)-algebras to the category of groups. If we factorize \( v = \prod_{l \in \Z/\mu} (t - \lambda_l) \) in \( K_v^{\text{sep}}[t] \) with \( \lambda_l = \lambda_{l+1} \), then we can write \( K_v^{\text{sep}}(t)_v = \prod_l K_v^{\text{sep}}((t - \lambda_l)) \) and \( \Psi_{ij} = (\Psi_{ijl})_l \) where \( \Psi_{ijl} \in K_v^{\text{sep}}((t - \lambda_l)) \). We set

\[
\Lambda_l := K(t)_v[\Psi_{11l}, \ldots, \Psi_{rrl}] \subset K^{\text{sep}}((t - \lambda_l))
\]
for each $l \in \mathbb{Z}/d$. Our main result in this paper is (see Lemma 4.16 and Theorems 4.14 and 5.15):

**Theorem 1.1.** The functor $\Gamma$ is representable by a smooth affine algebraic variety over $\mathbb{F}_q(t)_v$, also denoted by $\Gamma$. We have an equality $\dim \Gamma = \text{tr.deg}_{K(t)_v} A_l$ for each $l \in \mathbb{Z}/d$ and there exists a natural isomorphism $\Gamma \to \Gamma_M$ of affine group schemes over $\mathbb{F}_q(t)_v$.

This theorem is a $v$-adic analogue of Papanikolas’s Theorem 4.3.1 and 4.5.10 in [10], which treated $\infty$-adic objects. The proof of this theorem follows [10] closely, but since $K^{\text{sep}}(t)_v$ is not a field if $d > 1$, several arguments here are more complicated than in [10]. Let $K = \mathbb{F}_q(\theta)$ where $\theta$ is a variable independent of $t$. Papanikolas shows the equality of the transcendental degree of the field of periods (specialized at $t = \theta$) over $K$ and the dimension of the Tannakian Galois group using the so-called ABP-criterion proved by Anderson, Brownawell and Papanikolas in [2]. In fact he proved an algebraic independence of Carlitz logarithms. On the other hand, Anderson and Thakur [3] shows that the relation between the Carlitz zeta values and Carlitz logarithms. Then using these results, Chang and Yu [5] determined the all algebraic relations among the Carlitz zeta values. These applications are our motivation of this paper, but in this paper, we can only prove a $v$-adic analogue of the ABP-criterion for the rank one case.

In Section 3, first we review a theory of $\varphi$-modules in a general setting and construct a Tannakian category. In the $v$-adic case, we show that this category is equivalent to the category of Galois representations. In Section 4, we consider Frobenius equations in our situation, and construct $\Gamma$. In Section 5, we discuss the relation between $\Gamma$ and $\Gamma_M$, and prove that these are isomorphic in the $v$-adic case. This uses the fact that the set of $\mathbb{F}_q(t)_v$-valued points $\Gamma(\mathbb{F}_q(t)_v)$ is Zariski dense in $\Gamma$. Since $\Gamma(\mathbb{F}_q(t)_v)$ contains the Galois image, this is large enough in $\Gamma$. This is an essentially different point from Papanikolas’s proof for the $\infty$-adic case, in which the Zariski density is not proved and other facts are used to show this isomorphism. In Section 6, we discuss a $v$-adic analogue of the ABP-criterion. In Section 7, we prove the algebraic independence of certain “formal” polylogarithms.

**Acknowledgments.** The author thanks Yuichiro Taguchi for many helpful discussions on the contents of this paper and for reading preliminary manuscripts of this paper carefully.

## 2 Notations and terminology

### 2.1 Table of symbols

- $\mathbb{F}_q$ := the finite field of $q$ elements
- $\bar{k}$ := an algebraic closure of a field $k$
- $k^{\text{sep}}$ := the separable closure of a field $k$ in $\bar{k}$
- $\#S$ := the cardinality of a set $S$
- $\text{Mat}_{r \times s}(R)$ := the set of $r$ by $s$ matrices with entries in a ring or module $R$
- $\text{GL}_r(R)$ := the group of invertible $r$ by $r$ matrices with entries in a ring $R$
- $\text{Vec}(k)$ := the category of finite-dimensional vector spaces over a field $k$
- $\text{Rep}(G, R)$ := for a ring $R$ the category of finitely generated $R$-representations of an affine group scheme $G$ over $R$, or for a topological ring $R$ the category of finitely generated continuous $R$-representations of a topological group $G$
2.2 Action

Let $R$ be a ring or module and $f: R \to R$ a map. For a matrix $A = (A_{ij})_{ij} \in \text{Mat}_{r \times s}(R)$, we denote by $f(A)$ the matrix $(f(A_{ij}))_{ij}$.

Let $S$ be a set and $H$ a set of maps from $S$ to itself. Then we denote by $S^H$ the subset of $S$ consisting of elements which are fixed by $H$. For a map $f: S \to S$, we set $S^f := S^{\{f\}}$.

2.3 Base change

Let $R \to S$ be a homomorphism of commutative rings and $X$ a scheme over $R$. We denote by $X_S := X \times_{\text{Spec } R} \text{Spec } S$ the base change from $R$ to $S$ of $X$. We also denote by $X(S) := \text{Hom}_{\text{spec } R}(\text{Spec } S, X)$ the set of $S$-valued points of $X$ over $R$. When $R$ and $S$ are fields, we have a natural injection $X(R) \hookrightarrow X(S) \cong X_S(S)$. We always consider $X(R)$ as a subset of $X_S(S)$ via this injection.

3 \varphi-modules

3.1 \text{étale} \varphi-modules

In this subsection, we recall the definitions and properties of \text{étale} \varphi-modules (cf. [7]). Let $A$ be a commutative ring and $\sigma$ an endomorphism of $A$. For any $A$-module $M$, we put $M^{(\sigma)} := A \otimes_A M$, the scalar extension of $M$ by $\sigma$. A map $\varphi: M \to M$ is said to be \text{\sigma-semilinear} if $\varphi(x + y) = \varphi(x) + \varphi(y)$ and $\varphi(ax) = \sigma(a)\varphi(x)$ for all $x, y \in M$ and $a \in A$. Then it is clear that to give a \sigma-semilinear map $\varphi: M \to M$ is equivalent to giving an $A$-linear map $\varphi_\sigma: M^{(\sigma)} \to M$.

Definition 3.1. A \varphi-module $(M, \varphi)$ over $(A, \sigma)$ (or simply, $M$ over $A$) is an $A$-module $M$ endowed with a \text{\sigma-semilinear map} $\varphi: M \to M$. A morphism of \varphi-modules is an $A$-linear map which is compatible with the $\varphi$'s. When $A$ is a noetherian ring, a \varphi-module $(M, \varphi)$ is said to be \text{étale} if $M$ is a finitely generated $A$-module and $\varphi_\sigma: M^{(\sigma)} \to M$ is bijective.

We denote by $\Phi M_A$ the category of \varphi-modules over $A$ and $\Phi M_A^{\sigma}$ its full subcategory consisting of all \text{étale} \varphi-modules. For any \varphi-modules $M$ and $N$, we denote by $\text{Hom}_\varphi(M, N)$ the set of morphisms of $M$ to $N$ in $\Phi M_A$.

Let $A_\sigma[\varphi]$ be the ring (non commutative if $\sigma \neq \text{id}_A$) generated by $A$ and an element $\varphi$ with the relation

$$\varphi a = \sigma(a) \varphi$$

for each $a \in A$. Then it is clear that the category $\Phi M_A$ and the category of $A_\sigma[\varphi]$-modules are naturally identified. Hence, the category $\Phi M$ is an $A^\sigma$-linear abelian category.

For each \varphi-module $M$ and $N$, we denote by $M \otimes N$ the tensor product of $M$ and $N$, which is $M \otimes_A N$ as an $A$-module and has a $\varphi$-action defined by $\varphi \otimes \varphi$. Then the functor $\otimes$ is a bi-additive functor and $(A, \sigma)$ is an identity object in $\Phi M_A$ for this tensor product. Therefore the category $\Phi M_A$ is an abelian tensor category ([6]).

Proposition 3.2. There exists a natural isomorphism $A^\sigma \cong \text{End}_\varphi(A) := \text{Hom}_\varphi(A, A)$.

Proof. For any endomorphism $f \in \text{End}_\varphi(A)$, we have $\sigma(f(1)) = \varphi(f(1)) = f(\varphi(1)) = f(\sigma(1)) = f(1)$. Hence $f(1) \in A^\sigma$. Conversely for any element $a \in A^\sigma$, we have a map $f_a : A \to A; x \mapsto ax$. It is clear that $f_a \in \text{End}_\varphi(A)$. These are inverse to each other. \qed
Proposition 3.3. Assume that $A$ is noetherian and $\sigma$ is flat. Then the category $\Phi M_{A}^{\text{et}}$ is an abelian $A^\sigma$-linear tensor category.

Proof. It is clear that $\Phi M_{A}^{\text{et}}$ is closed under finite sums and tensor products, and the identity object $(A, \sigma)$ is étale. Therefore it is enough to show that for each étale $\varphi$-modules $M$ and $N$ and a morphism $f : M \rightarrow N$, the kernel and cokernel of $f$ in $\Phi M_{A}$ are étale. Since $M$ and $N$ are étale and $\sigma$ is flat, we have the commutative diagram

$$
\begin{array}{c}
0 \rightarrow (\ker f)^{(\sigma)} \rightarrow M^{(\sigma)} \rightarrow N^{(\sigma)} \rightarrow (\im f)^{(\sigma)} \rightarrow 0 \\
\varphi_{M,\sigma} \downarrow \quad \varphi_{N,\sigma} \downarrow \quad \varphi_{f,\sigma} \\
0 \rightarrow \ker f \rightarrow M \rightarrow N \rightarrow \im f \rightarrow 0,
\end{array}
$$

where $\varphi_{M,\sigma}$ and $\varphi_{N,\sigma}$ are isomorphisms and the rows are exact. Then we have that $\varphi_{f,\sigma}$ and $\varphi_{f,\sigma}''$ are isomorphism by a diagram chasing. \hfill \Box

Let $(M, \varphi_{M})$ and $(N, \varphi_{N})$ be $\varphi$-modules over $A$. If $\varphi_{M,\sigma} : M^{(\sigma)} \rightarrow M$ is an isomorphism, we define a $\varphi$-module $\text{Hom}(M, N)$, whose underlying $A$-module is the space $\text{Hom}_{A}(M, N)$ of $A$-module homomorphisms and a $\varphi$-action is defined by

$$
\text{Hom}_{A}(M, N)^{(\sigma)} \rightarrow \text{Hom}_{A}(M^{(\sigma)}, N^{(\sigma)}) \rightarrow \text{Hom}_{A}(M, N),
$$

where the first map is the natural map and the second map is defined by $f \mapsto \varphi_{N,\sigma} \circ f \circ \varphi_{M,\sigma}^{-1}$. There exists a natural morphism of $\varphi$-modules $\text{ev}_{M, N} : \text{Hom}(M, N) \otimes M \rightarrow N$. For each $M$ such that $\varphi_{M,\sigma}$ is an isomorphism, we set $M^\vee := \text{Hom}(M, A)$ the dual of $M$.

Proposition 3.4. Assume that $A$ is noetherian and $\sigma$ is flat. Then for any objects $M$ and $N$ in $\Phi M_{A}^{\text{et}}$, the $\varphi$-module $\text{Hom}(M, N)$ is étale, the contravariant functor

$$
\Phi M_{A}^{\text{et}} \rightarrow \text{Set}; \ T \mapsto \text{Hom}_{\varphi}(T \otimes M, N)
$$

is representable by $\text{Hom}(M, N)$ and $\text{ev}_{M, N}$ corresponds to $\text{id}_{\text{Hom}(M, N)}$.

Proof. Since $M$ and $N$ are finitely generated and $A$ is noetherian, $\text{Hom}(M, N)$ is also finitely generated. Since $\sigma$ is flat and $M$ is finitely presented, the map $\text{Hom}_{A}(M, N)^{(\sigma)} \rightarrow \text{Hom}_{A}(M^{(\sigma)}, N^{(\sigma)})$ is an isomorphism ([4], Chap. I, Sect. 2, Prop. 11). Since $\varphi_{M,\sigma}$ and $\varphi_{N,\sigma}$ are bijective, the $\varphi$-module $\text{Hom}(M, N)$ is étale. It is clear that there exists a natural isomorphism $\text{Hom}_{A}(T \otimes M, N) \cong \text{Hom}_{A}(T, \text{Hom}(M, N))$ which is functorial in $T$. Then we can calculate that the subspaces $\text{Hom}_{\varphi}(T \otimes M, N)$ and $\text{Hom}_{\varphi}(T, \text{Hom}(M, N))$ are corresponding with this isomorphism. The last assertion is clear. \hfill \Box

Proposition 3.5. Assume that $A$ is a field. Then the category $\Phi M_{A}^{\text{et}}$ is a rigid abelian $A^\sigma$-linear tensor category.

Proof. By Proposition 3.3, $\Phi M_{A}^{\text{et}}$ is an abelian $A^\sigma$-linear tensor category. By Proposition 3.4, $\Phi M_{A}^{\text{et}}$ has internal homs. Therefore it is enough to show that the natural map

$$
\otimes_{i \in I} \text{Hom}(M_{i}, N_{i}) \rightarrow \text{Hom}(\otimes_{i \in I} M_{i}, \otimes_{i \in I} N_{i})
$$

is an isomorphism for any finite families of objects $(M_{i})_{i \in I}$ and $(N_{i})_{i \in I}$, and the natural map

$$
M \rightarrow M^{\vee \vee}
$$

is an isomorphism for any object $M$ ([6]). These are true because $A$ is a field. \hfill \Box
3.2 $L$-triviality

Let $d$ be a positive integer and $F \subset E \subset L$ ring extensions where $F$, $E$ are fields and $L = \prod_{l \in \mathbb{Z}/d} L_l$ is a finite product of fields. For each $l$, we sometimes consider $L_l$ as a subset of $L$ in an obvious way. Let $\sigma : L \rightarrow L$ be a ring endomorphism. We assume that the triple $(F, E, L)$ satisfies the following properties:

- $\sigma(E) \subset E$ and $\sigma(L_l) \subset L_{l+1}$ for all $l$,
- $F = E^\sigma = L^\sigma$,
- $L$ is a separable extension over $E$.

Such a triple $(F, E, L)$ is called $\sigma$-admissible. See Lemma 3.24 for our main example. Another example can be found in [10].

Note that the separability of $L$ over $E$ is used to prove the smoothness of some algebraic groups (see Theorem 4.14), and not used in this section.

**Remark 3.6.** In [10], the term $\sigma$-admissible triple is defined only in the case where $L$ is a field and $\sigma$ is an isomorphism. Thus our general setting urges us to argue with greater care than in [10] at several points, and hence we decided not to avoid repeating similar arguments.

In this subsection, we consider $\varphi$-modules over $(E, \sigma|_E)$. For any $\varphi$-module $M$ over $E$, we set

$$V(M) := (L \otimes_E M)^\varphi$$

where $\varphi$ acts on $L \otimes_E M$ by $\sigma \otimes \varphi$. Then $V(M)$ is an $F$-vector space and $V$ forms a functor. We have natural maps

$$\iota_M : L \otimes_F V(M) \rightarrow L \otimes_E M,$$

$$\iota_{M, l} : L_l \otimes_F V(M) \rightarrow L_l \otimes_E M \text{ for all } l.$$ 

**Lemma 3.7.** Let $M$ be a $\varphi$-module, and let $\mu_1, \ldots, \mu_m \in V(M)$. If $\mu_1, \ldots, \mu_m$ are linearly independent over $F$, then they are linearly independent over $L$ (in $L \otimes_E M$).

*Proof.* Assume that the lemma is not true. Then there exist $m \geq 1$, $\mu_1, \ldots, \mu_m \in V(M)$ and $f_1, \ldots, f_m \in L$ such that, $\mu_1, \ldots, \mu_m$ are linearly independent over $F$, $(f_i)_l \neq 0$ and $\sum_i f_i \mu_i = 0$. We may assume that $m$ is minimal among the integers which satisfy the above properties. We also assume that $f_1 = (a_l)_l \in \prod_l L_l$ is non-zero. Let $a_{l_0} \neq 0$. Then there exists an element $f' \in L$ such that $f'f = e_{l_0}$, where $e_{l_0} \in L$ is the element such that the $l_0$-th component is one and the other components are zero. Then we have $\sum_i f' f_i \mu_i = 0$. Therefore we may assume that $f_1 = e_{l_0}$. Then we have

$$0 = \varphi(\sum_{j=1}^{d-1} (\sum_{i=1}^m f_i \mu_i) = \sum_{i=1}^m \sum_{j=0}^{d-1} \varphi^j(f_i) \mu_i = \sum_{i=1}^m (\sum_{j=0}^{d-1} \sigma^j(f_i)) \mu_i = \mu_1 + \sum_{i=2}^m (\sum_{j=0}^{d-1} \sigma^j(f_i)) \mu_i.$$ 

Therefore we may assume that $f_1 = 1$. Then we have

$$0 = \varphi(\sum_{i=1}^m f_i \mu_i) - \sum_{i=1}^m f_i \mu_i = \sum_{i=1}^m (\sigma(f_i) - f_i) \mu_i = \sum_{i=2}^m (\sigma(f_i) - f_i) \mu_i.$$ 

By the minimality of $m$, we have $f_i \in L^\sigma = F$ for all $i$. This contradicts the linear independence of $(\mu_i)_i$ over $F$. □
Corollary 3.8. For any \( \varphi \)-module \( M \), the maps \( \iota_M \) and \( \iota_{M,l} \) are injective and we have \( \dim_F V(M) \leq \dim_E M \).

\( \text{Proof.} \) By Lemma 3.7, \( \iota_M \) is injective. It is clear that \( \iota_M \) is injective if and only if \( \iota_{M,l} \) are injective for all \( l \). Therefore \( \iota_{M,l} \) is injective and we have an inequality \( \dim_F V(M) = \dim_{L_l}(L_l \otimes_F V(M)) \leq \dim_{L_l}(L_l \otimes_E M) = \dim_E M \). \( \square \)

Definition 3.9. Let \( M \) be a finite-dimensional \( \varphi \)-module over \( E \). We say that \( M \) is \( L\)-trivial if the map \( \iota_M \) is an isomorphism.

We denote by \( \Phi M^L_E \) the full subcategory of \( \Phi M_E \) consisting of all \( L\)-trivial \( \varphi \)-modules. Let \( M \) be a finite-dimensional \( \varphi \)-module over \( E \) and \( \mathbf{m} \in \text{Mat}_r(M) \) its \( E \)-basis. Then there exists a matrix \( \Phi \in \text{Mat}_{r \times r}(E) \) such that \( \varphi \mathbf{m} = \Phi \mathbf{m} \).

Proposition 3.10. The following conditions are equivalent:

1. \( M \) is \( L\)-trivial,
2. \( \iota_{M,l} \) is an isomorphism for each \( l \),
3. \( \iota_{M,l} \) is an isomorphism for some \( l \),
4. \( \dim_F V(M) = \dim_E M \),
5. there exists a matrix \( \Psi \in \text{GL}_r(L) \) such that \( \sigma \Psi = \Phi \Psi \).

\( \text{Proof.} \) It is clear that (1) \( \iff \) (2) \( \iff \) (3) \( \iff \) (4). Assume that the condition (4) is true. Then for each \( l \), we have \( \dim_{L_l}(L_l \otimes_F V(M)) = \dim_F V(M) = \dim_E M = \dim_{L_l}(L_l \otimes_E M) \). Therefore \( \iota_{M,l} \) is an isomorphism. This means that the condition (4) implies the condition (2).

Assume that the condition (1) is true. Let \( \mathbf{x} \) be an \( F \)-basis of \( V(M) \). Since the natural map \( \iota_M : L \otimes_F V(M) \to L \otimes_E M \) is an isomorphism, there exists a matrix \( \Psi \in \text{GL}_r(L) \) such that \( \Psi \mathbf{x} = 1 \otimes \mathbf{m} \). Then we have

\[
(\sigma \Psi) \mathbf{x} = (\sigma \Psi)(\varphi \mathbf{x}) = \varphi(\Psi \mathbf{x}) = \varphi(1 \otimes \mathbf{m}) = 1 \otimes \varphi \mathbf{m} = 1 \otimes \Phi \mathbf{m} = \Phi(1 \otimes \mathbf{m}) = \Phi \Psi \mathbf{x}.
\]

By Lemma 3.7, we have \( \sigma \Psi = \Phi \Psi \) and the condition (5) is true. Conversely, assume that the condition (5) is true. Then we have

\[
\varphi(\Psi^{-1}(1 \otimes \mathbf{m})) = (\sigma \Psi)^{-1}(1 \otimes \varphi \mathbf{m}) = (\Phi \Psi)^{-1}(1 \otimes \Phi \mathbf{m}) = \Psi^{-1}(1 \otimes \mathbf{m}).
\]

This means that \( \Psi^{-1}(1 \otimes \mathbf{m}) \in \text{Mat}_{r \times 1}(V(M)) \). Thus we have an inequality \( \dim_F V(M) \geq \dim_E M \) and the condition (4) is true. \( \square \)

Corollary 3.11. Let \( M \) be a finite-dimensional \( \varphi \)-module over \( E \). If \( M \) is \( L\)-trivial then \( M \) is étale.

\( \text{Proof.} \) By Proposition 3.10, there exists a matrix \( \Psi \in \text{GL}_r(L) \) such that \( \sigma \Psi = \Phi \Psi \). Since \( \sigma \) is injective, we have \( \det \Phi = \sigma(\det \Psi) \det \Psi^{-1} \neq 0 \). \( \square \)

Let \( M \) be an \( L\)-trivial \( \varphi \)-module over \( E \), \( \mathbf{m} \in \text{Mat}_{r \times 1}(M) \) an \( E \)-basis of \( M \) and \( \Phi \in \text{GL}_r(E) \) a matrix such that \( \varphi \mathbf{m} = \Phi \mathbf{m} \). By Proposition 3.10, there exists a matrix \( \Psi \in \text{GL}_r(L) \) such that \( \sigma \Psi = \Phi \Psi \).
Definition 3.12. The matrix $\Psi$ is called a period matrix of $M$ in $L$ or fundamental matrix of $\Phi$, and the entries of $\Psi$ are called periods of $M$ in $L$.

Note that $\Psi' \in \text{GL}_r(L)$ is another fundamental matrix of $\Phi$ if and only if $\Psi' = \Psi \delta$ for some $\delta \in \text{GL}_r(F)$. Indeed, if $\sigma(\Psi') = \Phi \Psi'$ then $\sigma(\Psi^{-1} \Psi') = \sigma(\Psi)^{-1} \sigma(\Psi') = (\Phi \Psi)^{-1} (\Phi \Psi') = \Psi^{-1} \Psi'$, hence $\Psi^{-1} \Psi' \in \text{GL}_r(L^\sigma) = \text{GL}_r(F)$, and the converse is clear.

Proposition 3.13. The period matrix $\Psi$ of $M$ is well-defined from $M$ as an element of $\text{GL}_r(E) \backslash \text{GL}_r(L) / \text{GL}_r(F)$.

Proof. Let $m' \in \text{Mat}_{r \times 1}(M)$ be another $E$-basis of $M$, $\Phi' \in \text{GL}_r(E)$ and $\Psi' \in \text{GL}_r(L)$ matrices such that $\varphi m' = \Phi' m'$ and $\sigma \Psi' = \Phi' \Psi'$. There exists a matrix $A \in \text{GL}_r(E)$ which satisfies $m' = Am$. Then we have $\varphi m' = \varphi(Am) = \varphi(A) \varphi m = \varphi(A) \Phi m = \sigma(A) \Phi A^{-1} m'$. Thus $\Phi' = \sigma(A) \Phi A^{-1}$. We also have $\sigma(\Psi) = \sigma(A) \sigma(\Psi) = \sigma(A) \Phi \Psi = \Phi'(\Psi)$. Hence we conclude that $\Psi' \in A \Psi \cdot \text{GL}_r(F)$.

Proposition 3.14. The set of entries of $\Psi^{-1}(1 \otimes m)$ forms an $F$-basis of $V(M)$.

Proof. By the proof of Proposition 3.10, we have that $\Psi^{-1}(1 \otimes m) \in \text{Mat}_{r \times 1}(V(M))$. Since $\dim_F V(M) = \dim_E M = r$, this is an $F$-basis of $V(M)$.

Proposition 3.15. The $\varphi$-module $(E, \sigma)$ is $L$-trivial.

Proof. We have equalities $V(E) = (L \otimes_E E)^\sigma = L^\sigma = F$. Therefore $\dim_F V(E) = 1 = \dim_E E$.

Proposition 3.16. Let $M$ and $N$ be $L$-trivial $\varphi$-modules. Then $M \oplus N$, $M \otimes N$ and $\text{Hom}(M, N)$ are also $L$-trivial.

Proof. Let $m \in \text{Mat}_{r \times 1}(E)$ be an $E$-basis of $M$, $\Phi_M \in \text{GL}_r(E)$ the matrix such that $\varphi m = \Phi_M m$ and $\Psi_M \in \text{GL}_r(L)$ a matrix which satisfies $\sigma \Psi_M = \Phi_M \Psi_M$. We also set $n \in \text{Mat}_{s \times 1}(E)$ an $E$-basis of $N$ and $\Phi_N \in \text{GL}_s(E)$, $\Psi_N \in \text{GL}_s(L)$ matrices which satisfy $\varphi n = \Phi_N n$ and $\sigma \Psi_N = \Phi_N \Psi_N$.

We set

$$m \oplus n := \begin{bmatrix} m \\ n \end{bmatrix}, \Psi_M \oplus \Psi_N := \begin{bmatrix} \Phi_M & 0 \\ 0 & \Phi_N \end{bmatrix} \quad \text{and} \quad \Psi_M \otimes \Psi_N := \begin{bmatrix} \Psi_M & 0 \\ 0 & \Psi_N \end{bmatrix}.$$ 

Then it is clear that $m \oplus n$ is an $E$-basis of $M \oplus N$, $\varphi(m \oplus n) = (\Phi_M \oplus \Phi_N)(m \oplus n)$ and $\sigma(\Psi_M \oplus \Psi_N) = (\Phi_M \oplus \Phi_N)(\Psi_M \oplus \Psi_N)$. Therefore $M \oplus N$ is $L$-trivial.

Set $m \otimes n$ to be an $E$-basis of $M \otimes N$ naturally obtained from $m$ and $n$. Let $\Phi_M \otimes \Phi_N$ be the Kronecker product of $\Phi_M$ and $\Phi_N$, and $\Psi_M \otimes \Psi_N$ be the Kronecker product of $\Psi_M$ and $\Psi_N$. Then it is clear that $\varphi(m \otimes n) = (\Phi_M \otimes \Phi_N)(m \otimes n)$ and $\sigma(\Psi_M \otimes \Psi_N) = (\Phi_M \otimes \Phi_N)(\Psi_M \otimes \Psi_N)$. Therefore $M \otimes N$ is $L$-trivial.

Let $m^\vee$ be the dual basis of $m$ for $M^\vee$. Then we have equalities $\varphi m^\vee = (\Phi_M^{-1})^t m^\vee$ and $\sigma(\Psi_M^{-1})^t = (\Phi_M^{-1})^t (\Psi_M^{-1})^t$, where $A^t$ is the transpose of a matrix $A$. Therefore $M^\vee$ is also $L$-trivial. Since $\Phi M^\vee_E$ is a rigid tensor category, we have an isomorphism $M^\vee \otimes N \cong \text{Hom}(M, N)$. Therefore $\text{Hom}(M, N)$ is $L$-trivial.

Proposition 3.17. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence in $\Phi M_E$. If $M$ is $L$-trivial, then $M'$ and $M''$ are also $L$-trivial.
Let \(L \otimes_F V(M) \to L \otimes_F V(M'')\) be the natural map and \(\iota'': \im(\kappa) \to L \otimes_E M''\) be the restriction of the map \(\iota_{M''}\). Then we have the commutative diagram
\[
\begin{array}{c}
0 \longrightarrow L \otimes_F V(M') \longrightarrow L \otimes_F V(M) \longrightarrow \im(\kappa) \longrightarrow 0 \\
\iota_M \downarrow \quad \iota_M \downarrow \quad \iota'' \downarrow \\
0 \longrightarrow L \otimes_E M' \longrightarrow L \otimes_E M \longrightarrow L \otimes_E M'' \longrightarrow 0,
\end{array}
\]
where the rows are exact, \(\iota_M\) is an isomorphism and \(\iota_M, \iota''\) are injective. Then we have that \(\iota_M\) and \(\iota''\) are isomorphism by a diagram chasing. Hence \(M'\) and \(M''\) are \(L\)-trivial.

**Proposition 3.18.** The category \(\Phi M^L_E\) is a rigid abelian \(F\)-linear tensor category.

**Proof.** By Proposition 3.5 and Corollary 3.11, it is enough to show that the category \(\Phi M^L_E\) is closed under direct sum, subquotient, tensor product and internal hom, and has an identity object for the tensor product. By Propositions 3.15, 3.16 and 3.17, these are true.

By Corollary 3.8, we can consider \(V\) as a functor from \(\Phi M^L_E\) to the category of finite-dimensional \(F\)-vector spaces \(\text{Vec}(F)\).

**Proposition 3.19.** The functor \(V : \Phi M^L_E \to \text{Vec}(F)\) is \(F\)-linear and exact.

**Proof.** It is clear that \(V\) is \(F\)-linear. Let \(0 \to M' \to M \to M'' \to 0\) be an exact sequence in \(\Phi M^L_E\). It is clear that \(0 \to V(M') \to V(M) \to V(M)\) is exact. This means that \(\dim_F V(M) \leq \dim_F V(M') + \dim_F V(M'')\). On the other hand, we have \(\dim_F V(M) = \dim_E M = \dim_E M' + \dim_E M'' \geq \dim_F V(M') + \dim_F V(M'')\).

**Proposition 3.20.** The functor \(V : \Phi M^L_E \to \text{Vec}(F)\) is faithful.

**Proof.** Let \(M\) and \(N\) be \(L\)-trivial \(\varphi\)-modules and \(\phi : M \to N\) a morphism in \(\Phi M_E\). Then we have an exact sequence
\[
0 \longrightarrow V(\ker \phi) \longrightarrow V(M) \longrightarrow V(N).
\]
Therefore if \(V(\phi) = 0\) then \(V(\ker \phi) = V(M)\). Since \(M\) is \(L\)-trivial, we have an inequality \(\dim_E \ker \phi \geq \dim F V(\ker \phi) = \dim F V(M) = \dim E M\). This means that \(\ker \phi = M\) and \(\phi = 0\).

**Proposition 3.21.** Let \(M\) and \(N\) be \(L\)-trivial \(\varphi\)-modules. Then there exists a natural isomorphism \(V(M) \otimes_F V(N) \to V(M \otimes N)\). The functor \(V : \Phi M^L_E \to \text{Vec}(F)\) is a tensor functor with respect to this isomorphism.

**Proof.** It is clear that there exists a natural isomorphism \((L \otimes_E M) \otimes_L (L \otimes_E N) \cong L \otimes_E (M \otimes N)\) which preserves \(\varphi\)-actions. By Lemma 3.7, the natural map \(V(M) \otimes_F V(N) \to (L \otimes_E M) \otimes_L (L \otimes_E N)\) is injective. Therefore we have a natural injection
\[
V(M) \otimes_F V(N) \hookrightarrow ((L \otimes_E M) \otimes_L (L \otimes E N))^\varphi \cong (L \otimes_E (M \otimes N))^\varphi = V(M \otimes N).
\]
Since \(\dim_F V(M) \otimes_F V(N) = \dim F V(M \otimes N)\), this map is a bijection. It is clear that this isomorphism is compatible with the associativity and the commutativity of tensor functors. It is also clear that \(V(E) = F\). Thus the functor \(V\) is a tensor functor ([6], Definition 1.8).
Recall that a *neutral Tannakian category* over a field $k$ is a rigid abelian $k$-linear tensor category $C$ for which $k \rightarrow \text{End}(1)$ and there exists an exact faithful $k$-linear tensor functor $\omega : C \rightarrow \text{Vec}(k)$, where $1$ is the unit object of $C$ ([6], Definition 2.19). Any such functor $\omega$ is said to be a fiber functor for $C$.

**Theorem 3.22.** The category $\Phi M^L_E$ is a neutral Tannakian category over $F$. The functor $V : \Phi M^L_E \rightarrow \text{Vec}(F)$ is an exact faithful $F$-linear tensor functor.

**Proof.** By Proposition 3.2, we have $\text{End}_\omega(E) \cong E^\omega = F$. By Proposition 3.18, the category $\Phi M^L_E$ is a rigid abelian $F$-linear tensor category. By Propositions 3.19, 3.20 and 3.21, the functor $V : \Phi M^L_E \rightarrow \text{Vec}(F)$ is an exact faithful $F$-linear tensor functor. $\square$

Let $M$ be an $L$-trivial $\varphi$-module over $E$. We set $T_M$ to be the Tannakian subcategory of $\Phi M^L_E$ generated by $M$, and $V_M : T_M \rightarrow \text{Vec}(F)$ the restriction of the functor $V$. We denote by $\Gamma_M$ the Tannakian Galois group of $(T_M, V_M)$. For any $F$-algebra $R$, we define the functor $V^R_M : T_M \rightarrow \text{Mod}(R)$ by $N \mapsto R \otimes_F V(N)$, where $\text{Mod}(R)$ is the category of finitely generated $R$-modules. Then by the definition of $\Gamma_M$, we have

$$\Gamma_M(R) = \text{Aut}^\circ(V^R_M)$$

where $\text{Aut}^\circ(V^R_M)$ is the group of invertible natural transformations from $V^R_M$ to itself which is compatible with the tensor products. Therefore we have a canonical injection $\Gamma_M(R) \hookrightarrow \text{GL}(R \otimes_F V(M))$.

### 3.3 $v$-adic case

Let $t$ be a variable and $v \in \mathbb{F}_q[t]$ a fixed monic irreducible polynomial of degree $d$. For any field $k$ containing $\mathbb{F}_q$, we set $k[t]_v := \lim_n (k[t]/v^n)$ and $k(t)_v := \mathbb{F}_q(t) \otimes_{\mathbb{F}_q[t]} k[t]_v$.

Let $\sigma$ be the ring endomorphism of $k[t]$

$$\sum a_it^i \mapsto \sum a_i^q \tau^i.$$ 

Then $\sigma$ naturally extends to an endomorphism of $k(t)_v$, also denoted by $\sigma$. Let $k'$ be a splitting field of $v$ over $k$, and we factorize $v = \prod_{l \in \mathbb{Z}/d}(t - \lambda_l)$ in $k'[t]$ with $\lambda'_l = \lambda_{l+1}$ for all $l \in \mathbb{Z}/d$. Then we have $k'(t)_v = \prod_{l \in \mathbb{Z}/d} k'(t - \lambda_l)$, and for any $a = (\sum a_{l+i}(t - \lambda_l)^l)_l \in k'(t)_v$,

$$\sigma(a) = (\sum a_{l-i}^q(t - \lambda_l)^{i})_l.$$ 

**Lemma 3.23.** For any field $k$ containing $\mathbb{F}_q$, we have $(k(t)_v)^\sigma = \mathbb{F}_q(t)_v$.

**Proof.** Clearly, $\mathbb{F}_q(t)_v = (\mathbb{F}_q(t)_v)^\sigma \subseteq (k(t)_v)^\sigma$. By the explicit description of the $\sigma$-action as above, we have $(k'(t)_v)^\sigma = \{(\sum a_{l+i}(t - \lambda_l)^l)_l \in \mathbb{F}_q(t)_v | a_{l+i}^q = a_{l+1+i}, \text{ for all } l \text{ and } i\}$. This set is isomorphic to $\mathbb{F}_q(t - \lambda_l)$ via the $l$-th projection for any $l$. On the other hand, we have $\mathbb{F}_q(t)_v \cong \mathbb{F}_q(t - \lambda_l)$. Thus the above inclusions are all equalities. $\square$

Fix a field $K$ containing $\mathbb{F}_q$ and assume that $K \cap \overline{\mathbb{F}_q} = \mathbb{F}_q$. Note that if $\mathbb{F}_q$ is not algebraically closed in $K$, then $K(t)_v$ may not be a field and the situation becomes more complicated. Thus in this paper, we always assume that $K \cap \overline{\mathbb{F}_q} = \mathbb{F}_q$.

**Lemma 3.24.** The triple $(\mathbb{F}_q(t)_v, K(t)_v, K^{sep}(t)_v)$ is $\sigma$-admissible.
Proof. Since \( v \) is irreducible in \( K[t] \), \( K(t)_v \) is a field. By Lemma 3.23, we have \( \mathbb{F}_q(t)_v = (K(t)_v)^\sigma = (K^{\text{sep}}(t)_v)^\sigma \). We need to check the separability. Fix an \( l \). We need to show that \( K^{\text{sep}}((t - \lambda_l)) / K(t)_v \) is a separable field extension. It is clear that \( K(t)_v = K'(t - \lambda_l) \) where \( K' = K(\lambda_l) \). On the other hand, \( K^{\text{sep}}((t - \lambda_l)) / K'(t - \lambda_l) \) is separable since \( K^{\text{sep}}/K' \) is separable (9, Exercise 26.2).

Let \( G_K := \text{Gal}(K^{\text{sep}}/K) \) be the absolute Galois group of \( K \). Then \( G_K \) acts on \( K^{\text{sep}}[t] \) in an obvious way. This action naturally extends to an action on \( K^{\text{sep}}(t)_v \). For each \( \tau \in G_K \) and \( a = (\sum_i a_i(t - \lambda_l)^i)_l \in \prod_l K^{\text{sep}}((t - \lambda_l)) \), we have

\[
\tau a = (\sum_i \tau a_{i+n}(t - \lambda_l)^i)_l,
\]

where \( n \in \mathbb{Z}/d \) is an element such that \( \tau|_{\mathbb{F}_q^d} = \sigma|_{\mathbb{F}_q^d}^{-n} \). It is clear that this action is compatible with \( \sigma \).

From now on, we consider \( \varphi \)-modules over the \( \sigma \)-admissible triple \((\mathbb{F}_q(t)_v, K(t)_v, K^{\text{sep}}(t)_v)\). Let \( M \) be an étale \( \varphi \)-module over \( K(t)_v \). The Galois group \( G_K \) acts on \( K^{\text{sep}}(t)_v \otimes M \) continuously by \( \tau \otimes \text{id} \) for each \( \tau \in G_K \). Since this action is compatible with \( \sigma \), the \( \mathbb{F}_q(t)_v \)-subspace \( V(M) \) is \( G_K \)-stable. We denote by \( V_K(M) \) this Galois representation. Conversely for any object \( V \) of \( \textbf{Rep}(G_K, \mathbb{F}_q(t)_v) \), we set

\[
D(V) := (K^{\text{sep}}(t)_v \otimes \mathbb{F}_q(t)_v, V)^{G_K},
\]

where \( G_K \) acts on \( K^{\text{sep}}(t)_v \otimes \mathbb{F}_q(t)_v, V \) by \( \tau \otimes \tau \) for \( \tau \in G_K \). Then we can define a \( \varphi \)-action on \( D(V) \) by \( \sigma \otimes \text{id} \).

Let \( M_0 \) be an étale \( \varphi \)-module over \( K[t]_v \). Then we can define an \( \mathbb{F}_q[t]_v \)-representation of \( G_K \)

\[
V_0(M_0) := (K^{\text{sep}}[t]_v \otimes K[t]_v, M_0)^\varphi,
\]

where \( \varphi \) acts on \( K^{\text{sep}}[t]_v \otimes K[t]_v, M_0 \) by \( \sigma \otimes \varphi \) and \( G_K \) acts on \( V_0(M_0) \) by \( \tau \otimes \text{id} \) for \( \tau \in G_K \). Conversely for any object \( T \) of \( \textbf{Rep}(G_K, \mathbb{F}_q[t]_v) \), we set

\[
D_0(T) := (K^{\text{sep}}[t]_v \otimes \mathbb{F}_q[t]_v, T)^{G_K},
\]

where \( G_K \) acts on \( K^{\text{sep}}[t]_v \otimes \mathbb{F}_q[t]_v, T \) by \( \tau \otimes \tau \) for \( \tau \in G_K \). Then we can define a \( \varphi \)-action on \( D_0(T) \) by \( \sigma \otimes \text{id} \).

**Theorem 3.25** ([8], Appendix). (1) For any étale \( \varphi \)-module \( M_0 \) over \( K[t]_v \), the natural map

\[
K^{\text{sep}}[t]_v \otimes \mathbb{F}_q[t]_v, V_0(M_0) \rightarrow K^{\text{sep}}[t]_v \otimes K[t]_v, M_0
\]

is bijective.

(2) For any \( \mathbb{F}_q[t]_v[G_K] \)-module \( T \) of finite type over \( \mathbb{F}_q[t]_v \), the natural map

\[
K^{\text{sep}}[t]_v \otimes K[t]_v, D_0(T) \rightarrow K^{\text{sep}}[t]_v \otimes \mathbb{F}_q[t]_v, T
\]

is bijective and the \( \varphi \)-module \( D_0(T) \) is étale.

(3) The functor \( V_0 : \Phi M^{\text{et}}_{K[t]_v} \rightarrow \textbf{Rep}(G_K, \mathbb{F}_q[t]_v) \) is a tensor equivalence, with a quasi-inverse \( D_0 : \textbf{Rep}(G_K, \mathbb{F}_q[t]_v) \rightarrow \Phi M^{\text{et}}_{\mathbb{F}_q[t]_v} \).

For any \( \varphi \)-module \( M_0 \) over \( K[t]_v \), we can define a \( \varphi \)-action on \( K(t)_v \otimes K[t]_v, M_0 \) by \( \sigma \otimes \varphi \).
Theorem 3.26. (1) A \( \varphi \)-module \( M \) over \( K(t)_v \) is \( K_{\text{sep}}(t)_v \)-trivial if and only if there exists a subspace \( M_0 \) of \( M \) which is an étale \( \varphi \)-module over \( K[t]_v \) such that \( M = K(t)_v \otimes_{K[t]_v} M_0 \).

(2) For any object \( V \) in \( \text{Rep}(G_K, \mathbb{F}_q(t)_v) \), the \( \varphi \)-module \( D(V) \) is \( \mathbb{F}_q(t)_v \)-trivial.

(3) The functor \( V : \Phi M_{K(t)_v}^{K_{\text{sep}}(t)_v} \rightarrow \text{Rep}(G_K, \mathbb{F}_q(t)_v) \) is a tensor equivalence, with a quasi-inverse \( D : \text{Rep}(G_K, \mathbb{F}_q(t)_v) \rightarrow \Phi M_{K(t)_v}^{K_{\text{sep}}(t)_v} \).

Proof. Let \( M \) be a \( \varphi \)-module over \( K(t)_v \) such that there exists a subspace \( M_0 \) which is an étale \( \varphi \)-module over \( K[t]_v \) and \( M = K(t)_v \otimes_{K[t]_v} M_0 \). Then by Theorem 3.25 (1), we have an isomorphism \( K_{\text{sep}}[t]_v \otimes_{\mathbb{F}_q[t]_v} V_0(M_0) \cong K_{\text{sep}}[t]_v \otimes_{K[t]_v} M_0 \). By tensoring \( K_{\text{sep}}(t)_v \) to the both sides of this isomorphism, we conclude that \( M \) is \( K_{\text{sep}}(t)_v \)-trivial.

Let \( V \) be an object in \( \text{Rep}(G_K, \mathbb{F}_q(t)_v) \). Then there exists a \( G_K \)-stable \( \mathbb{F}_q[t]_v \)-lattice \( T \) for \( V \). It is clear that \( D_0(T) \) is free over \( K[t]_v \) and \( D(V) = K(t)_v \otimes_{K[t]_v} D_0(T) \). Thus \( D(V) \) is \( K_{\text{sep}}(t)_v \)-trivial from the above argument and this proves (2). By Theorem 3.25 (2), we have an isomorphism \( K_{\text{sep}}(t)_v \otimes_{K(t)_v} D(V) \cong K_{\text{sep}}(t)_v \otimes_{\mathbb{F}_q(t)_v} V \). By taking the \( \varphi \)-fixed parts of the both sides of this isomorphism, we have an isomorphism \( V_K(D(V)) \cong V \).

Let \( M \) be a \( K_{\text{sep}}(t)_v \)-trivial \( \varphi \)-module over \( K(t)_v \). Then we have an isomorphism \( K_{\text{sep}}(t)_v \otimes_{\mathbb{F}_q(t)_v} V_K(M) \cong K_{\text{sep}}(t)_v \otimes_{K(t)_v} M \). By taking the \( G_K \)-fixed parts of the both sides of this isomorphism, we have an isomorphism \( D(V_K(M)) \cong M \), and this proves (3).

Therefore \( M \) comes from étale \( \varphi \)-module over \( K[t]_v \) and this proves (1).

4 Frobenius equations

Throughout this section, we fix a \( \sigma \)-admissible triple \( (F, E, L) \).

Example 4.1. The case \( (F, E, L) = (\mathbb{F}_q(t)_v, K(t)_v, K_{\text{sep}}(t)_v) \) is our main example of a \( \sigma \)-admissible triple, where the notation and the \( \sigma \)-action are as in Subsection 3.3.

Example 4.2. Let \( (F, E, L) = (\mathbb{F}_q(t)_v, K_{\text{rad}}(t)_v, K(t)_v) \) where \( K_{\text{rad}} := \cup_n K^{1/q^n} \), the maximal radical extension of \( K \) in \( \bar{K} \). The automorphism of \( K_{\text{rad}}(t) \)

\[
\sum_i a_it^i \mapsto \sum_i a_i^{1/q}t^i
\]

is naturally extends to an automorphism of \( \bar{K}(t)_v \). We define \( \sigma \) to be this action. Then \( (\mathbb{F}_q(t)_v, K_{\text{rad}}(t)_v, K(t)_v) \) is a \( \sigma \)-admissible triple. Note that, in this case we need to put \( L_l = \bar{K}(t - \lambda_{-l}) \). Note also that we do not use this type in this paper. However, the \( \sigma \)-action of this type is used in [10] and [5].

4.1 The group \( \Gamma \)

Let \( r \) be a positive integer. Fix matrices \( \Phi = (\Phi_{ij}) \in \text{GL}_r(E) \) and \( \Psi = (\Psi_{ij})_{i,j} \in \text{GL}_r(L) \) such that \( \Psi \) is a fundamental matrix for \( \Phi \). Thus we have an equation

\[
\sigma(\Psi) = \Phi \Psi.
\]

This means that the matrices \( \Phi \) and \( \Psi \) come from an \( L \)-trivial \( \varphi \)-module over \( E \). Since \( L = \prod_i L_i \), we can write \( \Psi_{ij} = (\Psi_{ijl}) \) for each \( i \) and \( j \). We set \( \Psi_l := (\Psi_{ijl})_{i,j} \in \text{GL}_r(L_l) \). Then we have \( \sigma(\Psi_l) = \Phi \Psi_{l+1} \) for all \( l \).
Let $X := (X_{ij})$ be an $r \times r$ matrix of independent variables $X_{ij}$, and set $\Delta := \det(X)$. We set $E[X, \Delta^{-1}] := E[X_{11}, X_{12}, \ldots, X_{rr}, \Delta^{-1}]$. Similarly $E[\Psi, \Delta(\Psi)^{-1}]$ and $E[\Psi_t, \Delta(\Psi_t)^{-1}]$ are defined. We define $E$-algebra homomorphisms $\nu : E[X, \Delta^{-1}] \to L$: $X_{ij} \mapsto \Psi_{ij}$ and $\nu : E[X, \Delta^{-1}] \to L_t$: $X_{ij} \mapsto \Psi_{ij}$. Set $p := \ker \nu, \Sigma := E[\Psi, \Delta(\Psi)^{-1}] \cong E[X, \Delta^{-1}] / p$. \(Z := \text{Spec } \Sigma, \; p_t := \ker \nu, \; \Sigma_t := E[\Psi_t, \Delta(\Psi_t)^{-1}] \cong E[X, \Delta^{-1}] / p_t\) and $Z_t := \text{Spec } \Sigma_t$. Then $Z_t$ are closed subschemes of $Z$ and $Z = \cup_t Z_t$. Let $\Lambda := \text{Frac}(\Sigma)$ and $\Lambda_t := \text{Frac}(\Sigma_t)$, the total rings of fractions.

Set $\Psi_1 := (\Psi_{ij} \otimes 1)_{i,j}, \Psi_2 := (1 \otimes \Psi_{ij})_{i,j}$ and $\tilde{\Psi} = (\Psi_{ij})^{-1}\Psi_2$ in $GL_r(L \otimes_E L)$. Since $L \otimes_E L = \prod_{i,m} L_t \otimes_E L_m$, we can write $\Psi_{ij} = (\tilde{\Psi}_{ijlm})_{i,m}$ with $\tilde{\Psi}_{ijlm} \in L_t \otimes_E L_m$ for each $i$ and $j$. We define $F$-algebra homomorphisms $\mu : F[X, \Delta^{-1}] \to L \otimes_E L_t: X_{ij} \mapsto \tilde{\Psi}_{ij}$ and $\mu_m : F[X, \Delta^{-1}] \to L_t \otimes_E L_m: X_{ij} \mapsto \tilde{\Psi}_{ijm}$. Set $q := \ker \mu, \; \Gamma := \text{Spec } F[X, \Delta^{-1}] / q$, $q_m := \ker \mu_m$ and $\Gamma_m := \text{Spec } F[X, \Delta^{-1}] / q_m$. Then $\Gamma_m$ are closed subschemes of $\Gamma$ and $\Gamma = \cup_m \Gamma_m$. By the next lemma, we can set $q_m := q_{0,m} = q_{1,m+1} = \cdots$ and $\Gamma_m := \Gamma_{0,m} = \Gamma_{1,m+1} = \cdots$.

**Lemma 4.3.** For any $l, m \in \mathbb{Z}/d$, we have $q_m = q_{l+1,m+1} = q_{l+2,m+2} = \cdots$.

**Proof.** Let $\bar{L}$ be the inductive limit of the inductive system $L \to L \to \cdots$, where the transition maps are $\sigma$. Then $L$ is a subring of $\bar{L}$ and $\sigma$ is naturally extends to an automorphism of $L$. We can define a $\sigma$-action on $\bar{L} \otimes_E \bar{L}$ by $\sigma \otimes \sigma$. This is an isomorphism and $L \otimes_E L$ is stable under this action. Thus we obtain an injective endomorphism $\sigma$ of $L \otimes_E L$. It is clear that $\sigma(L_t \otimes_E L_m) \subseteq L_{t+1} \otimes_E L_{m+1}$.

Write $\Psi_1 = (\Psi_{1,lm})_{l,m}$ and $\Psi_2 = (\Psi_{2,lm})_{l,m}$ with $\Psi_{1,lm} \in GL_r(L_t \otimes_E L_m)$, and set $\tilde{\Psi}_{lm} := (\tilde{\Psi}_{ijlm})_{i,j} \in GL_r(L_t \otimes_E L_m)$ for each $l$ and $m$. Then we obtain the equality $\sigma(\tilde{\Psi}_{lm}) = (\Phi \Psi_{1,l+1,m+1})^{-1}(\Phi \Psi_{2,l+1,m+1}) = \tilde{\Psi}_{l+1,m+1}$. For any $h(X) \in F[X, \Delta^{-1}]$, we have $h(\tilde{\Psi}_{lm}) = 0$ if and only if $h(\tilde{\Psi}_{l+1,m+1}) = 0$ since $\sigma(h(\tilde{\Psi}_{lm})) = h(\tilde{\Psi}_{l+1,m+1})$ and $\sigma$ is injective on $L \otimes_E L$. This proves the lemma. \(\square\)

For any $h(X) \in L[X, \Delta^{-1}]$, we denote by $h^\sigma(X)$ the polynomial obtained by applying $\sigma$ to the coefficients of $h(X)$. We define two endomorphisms

$$\sigma_0 : L[X, \Delta^{-1}] \to L[X, \Delta^{-1}]: h(X) \mapsto h^\sigma(X),$$

$$\sigma_1 : L[X, \Delta^{-1}] \to L[X, \Delta^{-1}]: h(X) \mapsto h^\sigma(\Phi X).$$

Then $\sigma_0(L_t[X, \Delta^{-1}]) \subseteq L_{t+1}[X, \Delta^{-1}]$ and $\sigma_1(L_t[X, \Delta^{-1}]) \subseteq L_{t+1}[X, \Delta^{-1}]$.

**Lemma 4.4.** We have $\sigma p_t \subseteq p_t, \; \sigma_1 p_t \subseteq p_{l+1}, \; \sigma_0 q = q, \; \sigma_0 q_m = q_m, \; \sigma^\nu = \nu \sigma_1|E[X, \Delta^{-1}]$ and $\sigma q_t = \nu_{l+1} \sigma_1|E[X, \Delta^{-1}]$ for each $l$ and $m$.

**Proof.** For any $h(X) \in E[X, \Delta^{-1}]$, we have $\nu_{l+1}(\sigma_1 h(X)) = \nu_{l+1}(\Phi X) = h^\sigma(\sigma \Psi_1 = \sigma h(\Psi_1)) = \sigma q_t h(X))$. If $h \in p_t$, then $\sigma h(\Psi_1) = \sigma h(\Psi_1) = 0$, and hence $\sigma_1 h \in p_{l+1}$. Since $q_t \subseteq F[X, \Delta^{-1}]$ and $\sigma_q|F[X, \Delta^{-1}]$ is id, we have $\sigma_0 q_t = q_t$. The other assertions are proved similarly. \(\square\)

For any ring homomorphism $R \to S$ and any ideal $a \subseteq R[X, \Delta^{-1}]$, we set $a_S := a \cdot S[X, \Delta^{-1}]$, the extension ideal of $a$.

**Lemma 4.5.** There exists a bijection between the set of ideals of $F[X, \Delta^{-1}]$ and the set of ideals of $L[X, \Delta^{-1}]$ which are $\sigma_0$-stable, via the extension and the restriction of ideals.
Proof. For any ideal \( a \subset F[X, \Delta^{-1}] \), it is clear that \( \sigma_0 a_L \subset a_L \). Because of the faithfully flatness of the inclusion \( F[X, \Delta^{-1}] \hookrightarrow L[X, \Delta^{-1}] \), we have \( a = e_L \cap F[X, \Delta^{-1}] \).

Conversely, we take any ideal \( b \subset L[X, \Delta^{-1}] \) with \( \sigma_0 b \subset b \), and set \( a := b \cap F[X, \Delta^{-1}] \). It is clear that \( b \supset a_L \); thus we need to show that the converse inclusion \( b \subset a_L \).

Take an \( F \)-basis \((g_i)_i \subset F[X, \Delta^{-1}] \) of \( F[X, \Delta^{-1}] \). Then this is an \( L \)-basis of \( L[X, \Delta^{-1}] \). For each \( h = \sum_i b_i g_i \in L[X, \Delta^{-1}] \), we set \( \text{supp}(h) := \{ i \in I | b_i \neq 0 \} \) and \( l(h) := \# \text{supp}(h) \). We take \( h \in b \) and show that \( h \in a_L \) by induction on \( l(h) \). If \( l(h) = 0 \), then \( h = 0 \in a_L \). Now suppose that \( l(h) > 0 \), and assume that if \( h \in b \) and \( l(h) < l(h) \) then \( h \in a_L \). Let \( e_l \in L \) be the element such that the \( l \)-th component is one and the other components are all zero. Then it is clear that \( \sigma_l e_l = e_{l+1} \). We write \( h = \sum_i b_i g_i \) and take \( i_l \) such that \( b_{i_l} \neq 0 \). Take \( l_0 \) such that the \( l_0 \)-th component of \( b_{i_l} \) is non-zero. Then there exists an element \( b' \in L \) such that \( b/b_{i_l} = e_{l_0} \). Since \( b \) is an ideal and \( \sigma_0 \)-stable, we have

\[
\begin{aligned}
b \supset \sum_{j=0}^{d-1} \sigma_j^0(b' h) &= \sum_{j=0}^{d-1} \sigma_j^0(b' \sum_i b_i g_i) = \sum_{i} \sum_{j=0}^{d-1} \sigma_j^0(b' b_i) g_i =: \sum_i c_i g_i =: h',
\end{aligned}
\]

Thus we assume that \( c_{i_l} \in L \setminus F \) for some \( i_l \) and \( d \geq 4 \). Since \( c_{i_l} = (c_{i_2, j_0}) \not\in F \), there exists an \( l_0 \) such that \( \sigma c_{i_2, j_0-1} \neq c_{i_2, j_0} \). Thus there is an element \( c' \in L \) such that \( (c'c_{i_2} - c_{i_2}) = e_{i_1} \). We set

\[
h' := \sum_i \sigma_i g_i := \sum_{j=0}^{d-1} \sigma_j^0(c'(\sigma c_i - c_i)) g_i = \sum_{j=0}^{d-1} \sigma_j^0(c'(\sigma_0 h' - h')) \in b.
\]

Then we have \( c_{i_1} = 0, c_{i_2} = 1 \) and \( \text{supp}(h'') \subset \text{supp}(h') \). For \( f = (f_i)_I \in L \), consider the element \( h := h' - fh'' = gi + (c_{i_2, l} - f_i)gi + \cdots \in b \). For any \( x = (x_i)_I \in L^X \), \( \sigma_x - x \in L^X \) if and only if \( \sigma x_l = x_{l+1} \) for all \( l \). Therefore, it is enough to take an element \( f \) such that \( c_{i_2, l} = f_l \) and \( \sigma(c_{i_2, l} - f_{l-1}) \neq c_{i_2, l} - f_l \) for all \( l \). Since \( \# L_l \geq 4 \), we can take \( (f_l)_I \) inductively so that \( f_1 \in X \setminus \{g_{i_1} \}, f_i \in L_l \setminus \{c_{i_2, l}, c_{i_2, l-1} - \sigma(c_{i_2, l-1} - f_{l-1}) \} \) for \( 2 \leq l < d \) and \( f_d \in L_d \setminus (\{c_{i_2, d}, c_{i_2, d} - \sigma(c_{i_2, d} - f_{d-1}) \} \cup \sigma逆(\sigma(c_{i_2, d} - c_{i_2, l} + f_l)) \). Then such \( (f_l)_I \) satisfies the above properties. Next, we show that \( h \in a_L \). Since \( \sigma_0 h - h \in b \) and \( l(\sigma_0 h - h) \leq l(h) \), we have \( \sigma_0 h - h \in a_L \) by induction hypothesis. Similarly, we can show that \( \sigma_0(a_{i_2}^{-1}h) - a_{i_2}^{-1}h \in a_L \). Therefore we have \( (\sigma(a_{i_2}^{-1}) - a_{i_2}^{-1})h = (\sigma(a_{i_2}^{-1}h) - a_{i_2}^{-1}h) - (\sigma(a_{i_2}^{-1}) - a_{i_2}^{-1})L^X, \) we have \( h \in a_L \).}

Lemma 4.6. The map \( \prod_1 b_i \mapsto (b_i)_I \) is a bijection between the set of ideals of \( L[X, \Delta^{-1}] \) which are \( \sigma_0 \)-stable, and the set of families \( (b_i)_I \) where \( b_i \) is an ideal of \( L_i[X, \Delta^{-1}] \) and \( \sigma_0 b_i \subset b_{i+1} \) for all \( i \).
Proof. This is clear. \qed

Lemma 4.7. For each \( l \), we give an ideal \( b_l \subset L_l[X, \Delta^{-1}] \) such that \( \sigma_0 b_l \subset b_{l+1} \). Then the restriction \( b_l \cap F[X, \Delta^{-1}] \) is independent of \( l \). The same is also true if we replace \( L_l \) by \( \Sigma_l \).

Proof. We only prove the case of \( L_l \). Let \( \ell \) be the natural injection \( F[X, \Delta^{-1}] \hookrightarrow L_l[X, \Delta^{-1}] \) and \( \pi_l \) the natural projection \( L[X, \Delta^{-1}] \rightarrow L_l[X, \Delta^{-1}] \) for each \( l \). For each \( l \), \( \sigma_0 \) induces a morphism \( L_l[X, \Delta^{-1}] \rightarrow L_{l+1}[X, \Delta^{-1}] \); we also denote this by \( \sigma_0 \). Then we have an equality \( \pi_{l+1} = \sigma_0 \pi_l \). For any \( h \in b_l \cap F[X, \Delta^{-1}] = (\pi_l)^{-1} b_l \), we have \( \pi_{l+1} (\ell(h)) = \sigma_0 (\pi_l (\ell(h))) \in \sigma_0 b_l \subset b_{l+1} \). Hence \( h \in (\pi_{l+1})^{-1} b_{l+1} = b_{l+1} \cap F[X, \Delta^{-1}] \). Therefore we obtain \( b_l \cap F[X, \Delta^{-1}] \subset b_{l+1} \cap F[X, \Delta^{-1}] \). Since the index set \( \mathbb{Z}/d \) is a finite cyclic group, this inclusion is an equality. \qed

For any ring \( R \), we denote by \( \text{GL}_{r/R} \) the \( R \)-group scheme of \( r \times r \) invertible matrices.

Proposition 4.8. (1) Let \( \phi : Z_L \rightarrow \text{GL}_{r/L} \) be the morphism of affine \( L \)-schemes defined by \( u \mapsto \Psi^{-1} u \) for any \( L \)-algebra \( S \) and any \( S \)-valued point \( u \in Z(S) \). Then \( \phi \) factors through an isomorphism \( \phi' : Z_L \rightarrow \Gamma_L \) of affine \( L \)-schemes.

(2) For any \( l \) and \( m \), let \( \phi_{lm} : Z_{m,L_l} \rightarrow \text{GL}_{r/L_l} \) be the morphism of affine \( L_l \)-schemes defined by \( u \mapsto \Psi^{-1} u \) for any \( L_l \)-algebra \( S \) and any \( S \)-valued point \( u \in Z_m(S) \). Then \( \phi_{lm} \) factors through an isomorphism \( \phi_{lm} : Z_{m,L_l} \rightarrow \Gamma_{m-l,L_l} \) of affine \( L_l \)-schemes.

Proof. We prove only (2). Then (1) can be proved by the same argument. We define two \( L_l \)-algebra homomorphisms:

\[
\begin{align*}
\alpha_l & : L_l[X, \Delta^{-1}] \rightarrow L_l[X, \Delta^{-1}], \quad X \mapsto \Psi^{-1} X, \\
\alpha_{lm} & : L_l[X, \Delta^{-1}] \rightarrow L_l[X, \Delta^{-1}] / \mathfrak{p}_{m,L_l} = L_l \otimes_{E} E[X, \Delta^{-1}] / \mathfrak{p}_m.
\end{align*}
\]

Then \( \phi_{lm} \) corresponds to \( \alpha_{lm} \) on the level of coordinate rings. Thus it is enough to show that \( \alpha_{l-1} \mathfrak{p}_{m,L_l} = \mathfrak{q}_{m-l,L_l} \).

For any \( h(X) \in L_l[X, \Delta^{-1}] \), we have

\[
\sigma_1 \alpha_l h = \sigma_1 (h(\Psi^{-1} X)) = h^{\sigma} (\sigma \Psi^{-1} \Phi X) = h^{\sigma} (\Psi^{-1} X) = \alpha_{l+1} h^{\sigma} (X) = \alpha_{l+1} \sigma_0 h.
\]

Therefore we have

\[
\alpha_{l+1} \sigma_1 = \sigma_0 \alpha_l^{-1}.
\]

Since \( \sigma_1 \mathfrak{p}_m \subset \mathfrak{p}_{m+1} \) by Lemma 4.4, we have an inclusion \( \sigma_0 \alpha_l^{-1} \mathfrak{p}_{m,L_l} = \alpha_{l+1}^{-1} \sigma_1 \mathfrak{p}_{m,L_l} \subset \alpha_{l+1}^{-1} \mathfrak{p}_{m+1,L_{l+1}} \). Replacing \( m \) by \( m + l \), we obtain \( \sigma_0 \alpha_l^{-1} \mathfrak{p}_{m+1,L_l} \subset \alpha_{l+1}^{-1} \mathfrak{p}_{m+1+1,L_{l+1}} \). We consider the family of ideals \( (\alpha_l^{-1} \mathfrak{p}_{m,L_l}) \). Then for each \( l \) and \( m \), we have \( (\alpha_l^{-1} \mathfrak{p}_{m+L_{l+1}} \cap F[X, \Delta^{-1}] \). Then \( l \) and \( m \), we have \( (\alpha_l^{-1} \mathfrak{p}_{m+L_{l+1}} \cap F[X, \Delta^{-1}] \) by Lemmas 4.5, 4.6 and 4.7. Again, replacing \( m \) by \( m - l \), we obtain an equality \( (\alpha_l^{-1} \mathfrak{p}_{m,L_l} \cap F[X, \Delta^{-1}] \).
We prove only (2). Then (1) can be proved by the same argument. Let (4.5, 4.6 and 4.7). Thus we obtain an inclusion
\[ p \subseteq \text{satisfies} \] coincides with \( \tilde{\mu} \). Therefore,
\[ q_{m-l} = q_{l,m} = \ker \mu_{m,l} = \alpha l^{-1}(F[X, \Delta^{-1}]) (\ker \tilde{\mu}) = \alpha l^{-1}(p, m, l) \cap F[X, \Delta^{-1}]. \]
Thus we have \( q_{m-l, l} = (\alpha l^{-1}(p, m, l) \cap F[X, \Delta^{-1}])_{l} = \alpha l^{-1}(p, m, l). \)

**Lemma 4.9.** (1) The ideal \( p \subseteq E[X, \Delta^{-1}] \) is maximal among the proper \( \sigma_{1} \)-invariant ideals.

(2) The family of ideals \( (p_{l})_{l} \) is maximal among the families of proper ideals \( (m_{l})_{l} \) of \( E[X, \Delta^{-1}] \) which satisfies \( \sigma_{1}m_{l} \subseteq m_{l+1} \) for all \( l \).

**Proof.** We prove only (2). Then (1) can be proved by the same argument. Let \( (m_{l})_{l} \) be a family of proper ideals of \( E[X, \Delta^{-1}] \) such that \( p_{l} \subseteq m_{l} \) and \( \sigma_{1}m_{l} \subseteq m_{l+1} \) for all \( l \). Let \( \alpha l \) be the homomorphism (4.1). We consider the family of ideals \( (\alpha l^{-1}m_{l}, l)_{l} \). Since \( \sigma_{0}\alpha l^{-1}m_{l, l} = \alpha l^{-1}\sigma_{1}m_{l, l} \subseteq \alpha l^{-1}m_{l+1, l, l+1} \), we can apply Lemma 4.6 to \( (\alpha l^{-1}m_{l}, l)_{l} \). Then \( \alpha l^{-1}m_{l, l} \cap F[X, \Delta^{-1}] \) is independent of \( l \), and we take a maximal ideal \( a \subseteq F[X, \Delta^{-1}] \) which contains this ideal. We also have \( (\alpha l^{-1}m_{l, l} \cap F[X, \Delta^{-1])}_{l} = \alpha l^{-1}m_{l, l} \) by Lemmas 4.5, 4.6 and 4.7. Thus we obtain an inclusion \( \alpha l^{-1}m_{l, l} \subseteq a_{l} \). We put \( M := F[X, \Delta^{-1}] / a \) and define a morphism
\[ \pi l : E[X, \Delta^{-1}] \hookrightarrow L[X, \Delta^{-1}] \twoheadrightarrow \rho_{l} \to L[X, \Delta^{-1}] / a_{l} \to L \otimes F M, \]
where \( \rho l \) is the natural projection and \( \beta l : L[X, \Delta^{-1}] / a_{l} \cong L \otimes F[X, \Delta^{-1}] / a = L \otimes F M. \) Then we have \( m_{l} \subseteq \ker \pi l \).

We define a \( \sigma \)-action on \( L \otimes F M \) by \( \sigma \otimes \text{id} \). In \( GL_{r}(L \otimes F M) \), we have
\[ \sigma_{l}(\pi l(X)) = \beta l_{l}(\rho l(\sigma_{l}^{-1}(X))) = \beta l_{l+1}(\rho l_{l+1}(\sigma_{0}(\sigma_{l}^{-1}(X)))) = \beta l_{l+1}(\rho l_{l+1}(\sigma l_{l+1}(\sigma_{1}(X)))) = \sigma_{l+1}(\sigma l_{l}(X)) = \sigma_{l+1}(\Phi X) = \Phi \sigma_{l+1}(X). \]
Set \( \pi(X) := (\pi l(X)) \in GL_{r}(L \otimes F M). \) Then we have \( \sigma(\pi(X)) = \Phi \pi(X). \) Since \( (L \otimes F M)^{\sigma} = M \), we obtain \( \delta := \pi(X)^{-1} \Phi \in GL_{r}((L \otimes F M)^{\sigma}) = GL_{r}(M) \). We define a \( \delta \)-action on \( (E \otimes F M)[X, \Delta^{-1}] \) by \( \delta \cdot h(X) := h(X\delta). \) We extend \( \pi l \) to
\[ \pi l' : (E \otimes F M)[X, \Delta^{-1}] = E[X, \Delta^{-1}] \otimes F M \to L \otimes F M. \]
Then we have \( p l \otimes F M \subseteq m l \otimes F M \subseteq \ker \pi l' = \delta \cdot \ker(\nu l \otimes \text{id} M) = \delta \cdot (p l \otimes F M), \) where the first equality is proved as follows: For any \( h(X) \in (E \otimes F M)[X, \Delta^{-1}] \), \( (\nu l \otimes \text{id} M)(h(X\delta^{-1})) = h(\Phi \Psi l^{-1}(\pi l(X))) = h(\pi l(X)) = h(\pi l'(X)) = \pi l'(h(X)). \) Thus \( h \in \ker \pi l' \) is equivalent to \( \delta^{-1} \cdot h \in \ker(\nu l \otimes \text{id} M). \) Since \( (E \otimes F M)[X, \Delta^{-1}] \) is a noetherian ring, \( p l \otimes F M \subseteq \delta \cdot (p l \otimes F M) \) implies \( p l \otimes F M = \delta \cdot (p l \otimes F M). \) Therefore we have \( p l \otimes F M = m l \otimes F M. \) Since \( (E \otimes F M)[X, \Delta^{-1}] \) is faithfully flat over \( E[X, \Delta^{-1}] \), we have \( p l = m l. \)
Lemma 4.10. (1) Let \( b \subset \Sigma[X, \Delta^{-1}] \) be an ideal which is \( \sigma_0 \)-invariant. Then we have
\( b = (b \cap F[X, \Delta^{-1}])_{\Sigma} \).

(2) Let \( b_l \subset \Sigma_l[X, \Delta^{-1}] \) be ideals which satisfy \( \sigma_0 b_l \subset b_{l+1} \) for all \( l \). Then we have
\( b_l = (b_l \cap F[X, \Delta^{-1}])_{\Sigma_l} \).

Proof. We prove only (2). Then (1) can be proved by the same argument. Let
\( a := b_l \cap F[X, \Delta^{-1}], \) which is independent of \( l \) by Lemma 4.7. Suppose that \( \sigma_0 a \subset b_{l+1} \) for some \( l \). Let \( (g_i)_{i \in I} \) be an \( F \)-basis of \( F[X, \Delta^{-1}] \) such that \( I = I_1 \cup I_2 \) and \( a = \bigoplus_{i \in I_1} F g_i \). Then we have
\( \Sigma_l[X, \Delta^{-1}] = \left( \bigoplus_{i \in I_1} \Sigma_l g_i \right) + \left( \bigoplus_{i \in I_2} \Sigma_l g_i \right) = \left( \bigoplus_{i \in I_1} \Sigma_l g_i \right) + a_{\Sigma_l} \).

Since \( a_{\Sigma_l} \subset b_{l+1} \), we can take a minimal finite set \( J \subset I_1 \) so that \( b_{l+1} \cap (\bigoplus_{i \in J} \Sigma_l g_i) \neq 0 \). By the injectivity of \( \sigma_0 \) and the inclusion \( \sigma_0 (b_l \cap \bigoplus_{i \in J} \Sigma_l g_i) \subset b_{l+1} \cap \bigoplus_{i \in J} \Sigma_l g_i \), \( J \) has the same properties for all \( l \). We fix \( j \in J \) and consider the ideal of \( \Sigma_l \cong E[X, \Delta^{-1}]/p_l \):
\[ m_l := \{ b \in \Sigma_l \mid \text{there exists } \sum_{i \in J} b_i g_i \in b_l \cap (\bigoplus_{i \in J} \Sigma_l g_i) \text{ such that } b_j = b \}. \]

Then \( m_l \) is a non-zero ideal by the minimality of \( J \), and it is clear that \( \sigma_0 m_l \subset m_{l+1} \). By Lemma 4.4 we have \( \sigma m_l = \nu_{l+1} \sigma_0 m_l \). Hence we can apply Lemma 4.9 to the inverse image of \( (m_l)_l \) in \( E[X, \Delta^{-1}] \). Therefore we have \( m_l = \Sigma_l \). Thus for each \( l \), there exists an element \( h_l = \sum_{i \in J} b_i g_i \in b_l \cap (\bigoplus_{i \in J} \Sigma_l g_i) \) such that \( b_j = 1 \). Then we have \( b_{l+1} \cap \sigma_0 h_l = h_{l+1} = \sum_{i \in J} (\sigma h_i - b_{l+1,i}) g_i \). By the minimality of \( J \), \( \sigma h_i = b_{l+1,i} \) for all \( i \) and \( l \). We put \( b_i := (h_i)_{i \in I_1} \cap \Sigma_l = \bigoplus_{i \in J} \Sigma_l g_i \). Then we have \( \sigma h_i = (\sigma b_{l+1,i}) g_i = h_i g_i \). Hence \( b_i \in F_l \) and \( b_i \in F_l \) via the \( l \)-th projection. Then \( 0 \neq h_l = \sum_{i \in J} b_i g_i \in b_l \cap F[X, \Delta^{-1}] = a = \bigoplus_{i \in I_1} F g_i \).

This contradicts \( J \cap I_1 = \emptyset \).

\[ \square \]

Proposition 4.11. (1) Let \( \psi : Z \times_E \Sigma \to \Sigma \times E \) be the morphism of affine \( E \)-schemes defined by \( (u, v) \mapsto (u, u^{-1}v) \) for any \( E \)-algebra \( S \) and any \( S \)-valued point \( (u, v) \in Z(S) \times Z(S) \). Then \( \psi \) factors through an isomorphism \( \psi' : Z \times_E Z \to Z \times \Sigma \).

(2) For any \( l \) and \( m \), let \( \psi_{lm} : Z_l \times_E Z_{l+m} \to Z_l \times_E \Sigma \) be the morphism of affine \( E \)-schemes defined by \( (u, v) \mapsto (u, u^{-1}v) \) for any \( E \)-algebra \( S \) and any \( S \)-valued point \( (u, v) \in Z_l(S) \times Z_{l+m}(S) \). Then \( \psi \) factors through an isomorphism \( \psi_{lm}' : Z_l \times E Z_{l+m} \to Z_l \times E \Sigma_l \).

Proof. We prove only (2). Then (1) can be proved by the same argument. Let \( \alpha_l \) and \( \alpha_{lm} \) be the homomorphisms (4.1) and (4.2). We restrict the domain of \( \alpha_{lm} \) to \( \Sigma_l[X, \Delta^{-1}] \cong E[X, \Delta^{-1}]/p_l \otimes E[X, \Delta^{-1}] \) and the target of \( \alpha_{lm} \) to \( \Sigma_l \otimes_E E[X, \Delta^{-1}]/p_m \cong E[X, \Delta^{-1}]/p_l \otimes E[X, \Delta^{-1}]/p_m \). Then \( \psi_{lm} \) corresponds to \( \alpha_{lm} \) on the level of coordinate rings. Hence it is enough to show that \( \alpha_{lm}^{-1} p_{m+l, \Sigma_l} = q_{m, \Sigma_l} \).

Since we have an inclusion \( \sigma_0 \alpha_{lm}^{-1} p_{m+l, \Sigma_l} = \alpha_{lm}^{-1} \sigma_0 p_{m+l, \Sigma_l} \subset \alpha_{lm}^{-1} p_{m+l, \Sigma_l} \cap F[X, \Delta^{-1}] \) is independent of \( l \) by Lemma 4.7, and we can apply Lemma 4.10 to \( (\alpha_{lm}^{-1} p_{m+l, \Sigma_l})_l \). Then we have \( \alpha_{lm}^{-1} p_{m+l, \Sigma_l} = (\alpha_{lm}^{-1} p_{m+l, \Sigma_l} \cap F[X, \Delta^{-1}])_{\Sigma_l} = \)
\(a_m, \Sigma_i\). On the other hand we have \(a_l^{-1}q_{m+l, L_l} = q_{m, L_l}\) by Proposition 4.8. Therefore \(q_{m, L_l} = q_{m+l, L_l} = a_l^{-1}(q_{m+l, \Sigma_l}) L_l = (a_l^{-1}q_{m+l, \Sigma_l}) L_l = a_m, \Sigma_l). Then we have \(q_m = a_m\) since \(L_l[X, \Delta^{-1}] = \) faithfully flat over \(F[X, \Delta^{-1}]\). Thus we obtain \(a_l^{-1}p_{m+l, L_l} = a_m, \Sigma_l = q_{m, \Sigma_l} \).

The next lemma is proved by an elementary argument. Thus we omit the proof. This lemma is applied to the \(S\)-valued points of the diagrams in Proposition 4.11 where \(S\) is an \(E\)-algebra.

**Lemma 4.12.** (1) Let \(G\) be a group, \(A\) and \(B\) be non-empty subsets of \(G\) such that the map

\[\psi : A \times A \to A \times G; (u, v) \mapsto (u, u^{-1}v)\]

factors through a bijection \(\psi' : A \times A \to A \times B\). Then \(B\) is a subgroup of \(G\), \(A\) is stable under right-multiplication by elements of \(B\) and \(A\) becomes a \(B\)-torsor.

(2) Let \(G\) be a group, \(A_l\) and \(B_m\) be non-empty subsets of \(G\) such that the map

\[\psi_{lm} : A_l \times A_l + m \to A_l \times G; (u, v) \mapsto (u, u^{-1}v)\]

factors through a bijection \(\psi'_{lm} : A_l \times A_l + m \to A_l \times B_m\) for each \(l\) and \(m\). Then \(B_0\) is a subgroup of \(G\), \(A_l\) is stable under right-multiplication by elements of \(B_0\) and \(A_l\) becomes a \(B_{0, l}\)-torsor for each \(l\). Moreover, for any \(u \in A_l\) and \(v \in A_l + m\) there exists an element \(y \in B_m\) such that \(v = uy\). The multiplication in \(G\) induces maps \(A_l \times B_m \to A_l + m\) and \(B_m \times B_m \to B_{m, m'}\), and the inversion in \(G\) induces a map \(B_{m} \to B_{m, 0}\).

By Proposition 4.11 and Lemma 4.12, we have surjective maps

\[(4.3) \quad Z_l(S) \times Z_{l+m}(S) \to \Gamma_m(S); (u, v) \mapsto u^{-1}v,\]

\[(4.4) \quad Z_l(S) \times \Gamma_m(S) \to Z_{l+m}(S); (x, y) \mapsto xy\]

for any \(E\)-algebra \(S\).

**Theorem 4.13.** (1) The \(F\)-scheme \(\Gamma\) is a closed \(F\)-subgroup scheme of \(GL_{r/F}\), the \(E\)-scheme \(Z\) is stable under right multiplication by \(\Gamma_E\) and is a \(\Gamma_{E, l}\)-torsor.

(2) The \(F\)-scheme \(\Gamma_0\) is a closed \(F\)-subgroup schemes of \(GL_{r/F}\), the \(E\)-scheme \(Z_l\) is stable under right multiplication by \(\Gamma_{0, E}\) and is a \(\Gamma_{0, E, l}\)-torsor for each \(l\).

(3) The \(F\)-scheme \(\Gamma_m\) is stable under right and left multiplications by \(\Gamma_0\) and is a \(\Gamma_{0, l}\)-torsor for each \(m\).

**Proof.** We prove only (2). Then (1) and (3) can be proved by the same argument. By Proposition 4.11, we have a bijection \(Z_l(S) \times Z_{l+m}(S) \to Z_l(S) \times \Gamma_{0, E}(S); (u, v) \mapsto (u, u^{-1}v)\) for any \(E\)-algebra \(S\). Since \(Z_l(S)\) is non-empty, Lemma 4.12 implies that \(\Gamma_{0, E}\) is a closed subgroup scheme of \(GL_{r/E}\) and \(Z_l/E\) is a \(\Gamma_{0, E, l}\)-torsor. Therefore \(\Gamma_0\) is a closed subgroup scheme of \(GL_{r/F}\) by the faithfully flatness of the inclusion \(F \to E\). Similarly, \(Z_l\) is a \(\Gamma_{0, E, l}\)-torsor by the faithfully flatness of the inclusion \(E \to \bar{E}\).

**Theorem 4.14.** (a) The \(E\)-schemes \(Z\) and \(Z_l\) are smooth.

(b) The \(F\)-schemes \(\Gamma\) and \(\Gamma_m\) are smooth.

(c) If \(E\) is algebraically closed in the fraction field \(\Lambda_{l_0}\) of \(\Sigma_{l_0}\) for some \(l_0\), then \(Z_l\) and \(\Gamma_m\) are absolutely irreducible.

(d) \(\dim \Gamma = \dim \Gamma_m = \text{tr.deg}_E \Lambda_l\).
Proof. (a), (b) Since $L_l/E$ is a separable extension, $\Lambda/E$ is also a separable extension, where $\Lambda = \text{Frac}(\Sigma)$ the total ring of fractions of $\Sigma$. Thus for any field extension $\Omega/E$, $\Lambda \otimes_E \Omega$ is reduced. Therefore $\Sigma \otimes_E \Omega$ is reduced and $Z = \text{Spec} \Sigma$ is absolutely reduced. Since $\Gamma_E \cong Z_E$, $\Gamma$ is absolutely reduced. Since $\Gamma$ is an algebraic group, the property that $\Gamma$ is absolutely reduced implies that $\Gamma$ is smooth. Again since $\Gamma_E \cong Z_E$, we have that $Z$ is smooth. The statements of Lemma 4.16.

(c) For any field extension $\Omega/E$, $\Lambda_0 \otimes_E \Omega$ is an integral domain by the assumption. Therefore $\Sigma_0 \otimes_E \Omega$ is an integral domain and $Z_0$ is absolutely integral. Since $Z_{l,E} \cong \Gamma_{m,E}$ for all $l$ and $m$, $Z_l$ and $\Gamma_m$ are all absolutely integral.

(d) We have an equality $\dim \Gamma = \dim \Gamma_m = \dim \Gamma_0 = \dim Z_l = \text{tr.deg}_E \Lambda_l$. \hfill \Box

Corollary 4.15. (1) There exists a divisor $d'$ of $d$ such that if $l \equiv l' \pmod{d'}$ then $Z_l = Z_{l'}$ and if $l \not\equiv l' \pmod{d'}$ then $Z_l \cap Z_{l'} = \emptyset$.

(2) If $m \equiv m' \pmod{d'}$ then $\Gamma_m = \Gamma_{m'}$ and if $m \not\equiv m' \pmod{d'}$ then $\Gamma_m \cap \Gamma_{m'} = \emptyset$.

Therefore we can write $Z = \prod_{l \in \mathbb{Z}/d'} Z_l$, $\Sigma = \prod_{l \in \mathbb{Z}/d'} \Sigma_l$, $\Lambda = \prod_{l \in \mathbb{Z}/d'} \Lambda_l$ and $\Gamma = \prod_{m \in \mathbb{Z}/d'} \Gamma_m$.

Proof. (1) Since $Z_l$ is a $\Gamma_0,E$-torsor and absolutely reduced for all $l$, it is clear that $Z_l = Z_{l'}$ or $Z_l \cap Z_{l'} = \emptyset$. We have the surjective map (4.4) : $Z_l(\bar{E}) \times \Gamma_1(\bar{E}) \rightarrow Z_{l+1}(\bar{E})$. Therefore if $Z_l = Z_{l'}$, then $Z_{l+1}(\bar{E}) = Z_{l+1}(\bar{E})$. Hence if we take $d'$ to be the minimum positive integer such that $Z_0 = Z_{d'}$, then $d'$ satisfies the desired properties.

(2) By the same argument of the proof of (1), there exists a divisor $d''$ of $d$ which is the period of $(\Gamma_m)_m$. Then by the map (4.3), we have

$$\Gamma_m(E) = Z_l(\bar{E})^{-1} Z_{l+m}(\bar{E}) = Z_l(\bar{E})^{-1} Z_{l+m+d'}(\bar{E}) = \Gamma_{m+d'}(\bar{E}).$$

This means that $d''|d'$. By the map (4.4), we have

$$Z_l(\bar{E}) = Z_l(\bar{E}) \Gamma_0(\bar{E}) = Z_l(\bar{E}) \Gamma_{d'}(\bar{E}) = Z_{l+d'}(\bar{E}).$$

This means that $d'|d''$. \hfill \Box

4.2 $\Gamma$-action

For any $F$-algebras $R$ and $S$, we set $S^{(R)} := R \otimes_F S$. In particular, if $R = F'$ is a field, we set $S' := S^{(F')}$. If $\sigma$ acts on $S$, we define the $\sigma$-action on $S^{(R)}$ by $\text{id} \otimes \sigma$. Note that, if $S' = F$, then we have $(S^{(R)})^\sigma = R$. Let $\text{Aut}_\sigma(\Sigma^{(R)}/E^{(R)})$ denote the group of automorphisms of $\Sigma^{(R)}$ over $E^{(R)}$ that commute with $\sigma$. Similarly we define $\text{Aut}_\sigma(\Lambda^{(R)}/E^{(R)})$.

For any $\gamma \in \Gamma(R)$, we obtain an automorphism $Z_{E(R)} \rightarrow Z_{E(R)}; x \mapsto x\gamma$. On the level of coordinate rings, this corresponds to an automorphism $\Sigma^{(R)} \rightarrow \Sigma^{(R)}; h(\Psi) \mapsto \gamma(h(\Psi)) := h(\Psi\gamma)$. Note that $\Sigma^{(R)} = E^{(R)} \otimes_F \Sigma = E^{(R)}[\Psi, \Delta(\Psi)^{-1}] \cong E^{(R)}[X, \Delta, \Delta^{-1}]/p_{E(R)}$. Thus we have a group homomorphism $\kappa_R: \Gamma(R) \rightarrow \text{Aut}(\Sigma^{(R)}/E^{(R)})$.

Lemma 4.16. (1) For any $F$-algebra $R$, the map $\kappa_R$ induces an isomorphism $\Gamma(R) \cong \text{Aut}_\sigma(\Sigma^{(R)}/E^{(R)})$. Its inverse is the map $\alpha \mapsto \Psi^{-1}(\alpha \Psi_i)_{ij}$.

(2) If $\Lambda / E$ is a regular extension (i.e. separable extension and $F$ is algebraically closed in $\Lambda$) for all $l$ and $F'/F$ is an algebraic extension of fields, then we have $\text{Aut}_\sigma(\Sigma'/E') \cong \text{Aut}_\sigma(\Lambda'/E')$. 19
Proof. (1) For any \( \gamma \in \Gamma(R) \) and \( h(\Psi) \in \Sigma^{(R)} \), we have \( \sigma(\gamma.h(\Psi)) = \sigma(h(\Psi)\gamma) = h^\sigma(\sigma(\Psi)\gamma) = h^\sigma(\Phi\gamma) = \gamma.(h^\sigma(\Phi)) = \gamma.(\sigma(h(\Psi))) \). Hence \( \kappa_R(\gamma) \) commutes with \( \sigma \). Suppose that \( \kappa_R(\gamma) \) is the identity. Then \( h(\Psi) = h(\Psi) \) for any \( h(\Psi) \in \Sigma^{(R)} \). In particular if we take \( h(\Psi) = \Psi_{ij} \) for each \( i \) and \( j \), then we obtain \( \Psi = \Psi \) in \( \text{GL}_r(\Sigma^{(R)}) \). Therefore \( \gamma = 1 \) and this means that \( \kappa_R \) is injective. Conversely, let \( \alpha \in \text{Aut}_\sigma(\Sigma^{(R)}/E^{(R)}) \) be any element. Then \( \alpha \) corresponds to an automorphism \( \tilde{\alpha} : Z_{E^{(R)}} \rightarrow Z_{E^{(R)}} \), and \( \tilde{\alpha} \) maps the \( \Sigma^{(R)} \)-valued point \( \Psi \) to \((\alpha \Psi_{ij})_{ij} \). By Theorem 4.13, there exists an element \( \gamma \in \Gamma(\Sigma^{(R)}) \) such that \( \Psi \gamma = (\alpha \Psi_{ij})_{ij} \). Then for any \( h(\Psi) \in \Sigma^{(R)} \), we have \( \alpha(h(\Psi)) = h((\alpha \Psi_{ij})_{ij}) = h(\Psi) \gamma \). Thus we obtain \( \sigma(\gamma.h(\Psi)) = \sigma(\alpha(h(\Psi))) = \alpha(\sigma(h(\Psi))) = \alpha(h^\sigma(\Phi)) = h^\sigma(\Phi) = h^\sigma(\Phi) \gamma \). On the other hand, \( \sigma(\gamma.h(\Psi)) = \sigma(h(\Psi)_{\gamma}) = h^\sigma((\sigma(\Phi)_{\gamma})) = \sigma(h(\Phi)_{\gamma}) \). If we take \( h(\Psi) = \Psi_{ij} \) for each \( i \) and \( j \), we obtain \( \Phi \Psi(\sigma) = \Phi \Psi \gamma \). Hence \( \sigma = \gamma \). Therefore we have \( \gamma \in \Gamma(R) \) and \( \kappa_R(\gamma) = \alpha \).

(2) Since \( \Lambda = \text{Frac}(\Sigma) \), any automorphism of \( \Sigma \) extends uniquely to an automorphism of \( \Lambda \). Conversely if \( \alpha \in \text{Aut}_\sigma(\Lambda/E) \), then \( \sigma(\alpha(\Psi)) = \alpha(\sigma(\Psi)) = \alpha(\Phi) = \Phi \cdot \alpha(\psi) \) in \( \text{GL}_r(\Lambda) \). Thus we have \( \alpha(\Psi) \in \Psi \cdot \text{GL}_r(\Lambda) \). This implies that \( \alpha(\Sigma) \subset \Sigma \). Similarly, we have \( \alpha^{-1}(\Sigma) \subset \Sigma \). Therefore \( \alpha(\Sigma) = \Sigma \).

(3) Since \( \Lambda^0/F \) is a regular extension, so is \( E/F \). Since \( F'/F \) is an algebraic extension, \( E' \) and \( \Lambda'_i \) are fields and \( \Lambda' = \text{Frac}(\Sigma') \). Therefore \( \Lambda' = \prod_{i \in E'/F} \Lambda'_i \) is a finite product of fields and \( \Lambda' = \text{Frac}(\Sigma') \). Then, the proof is the same as (2).

We prepare some lemmas about Zariski density.

**Lemma 4.17.** Let \( \Omega/k \) be a field extension such that \( \Omega \) is an algebraically closed field, \( X \) an algebraic variety over \( k \) and \( Y \) a closed subvariety of \( X_\Omega \). If \( X(k) \cap Y(\Omega) \) is Zariski dense in \( Y \), then \( Y \) is defined over \( k \), i.e. there exists some algebraic variety \( Y_0 \) over \( k \) such that \( Y = Y_0, \Omega \).

**Proof.** We may assume that \( X \) is affine. Let \( k[X] \) be the coordinate ring of \( X \) and \( \Omega[k[X]] \) the coordinate ring of \( X_\Omega \). Then we have \( \Omega[X] = \Omega \otimes_k k[X] \). Let \( a \subset \Omega[X] \) be the defining ideal of \( Y \) and \( a_\Omega := a \cap k[X] \). We need to show that \( a_\Omega \in \Omega \). Thus we assume that \( a_\Omega \subset \Omega \). Let \( (g_i)_{i \in I} \) be a \( k \)-basis of \( k[X] \) such that \( (g_i)_{i \in I} \) is a \( k \)-basis of \( a_k \) for some \( I \subset I' \). We also take \( (c_j)_{j \in J} \) to be a \( k \)-basis of \( \Omega \). Since \( a_\Omega \subset \Omega \), there exists a non-zero element \( f = \sum_{i \in I} a_i g_i \in a \) where \( a_i \in \Omega \). Write \( a_i = \sum_j c_{ij} a_j \) (\( c_{ij} \in k \)). Then we can write \( f = \sum_{i \in I} c_{ij} a_i g_i \). For any \( x \in X(k) \cap Y(\Omega) \), we have \( \sum_{j} c_{ij} \sum_{i \in I \setminus I} \alpha_i g_i(x) = f(x) = 0 \). Since \( (c_j)_{j} \) is linearly independent over \( k \), we have \( \sum_{i \in I \setminus I} \alpha_i g_i(x) = 0 \) for all \( j \). By the density assumption, we obtain \( \sum_{i \in I \setminus I} \alpha_i g_i(x) = 0 \) for all \( x \in Y(\Omega) \). Therefore \( \sum_{i \in I \setminus I} \alpha_i g_i \in a \cap k[X] = a_k = \oplus_{i \in I_k} k g_i \) for all \( j \). Since \( (g_i)_{i \in I} \) is linearly independent over \( k \), we have \( \alpha_i = 0 \). Thus \( f = 0 \), which is a contradiction.

**Corollary 4.18.** Let \( \Omega/k \) be a field extension such that \( \Omega \) is an algebraically closed field and \( X \) an algebraic variety over \( k \). If \( X(k) \) is Zariski dense in \( X \), then \( X(k) \) is Zariski dense in \( X_\Omega \).

**Proof.** We take \( Y \) to be the Zariski closure of \( X(k) \) in \( X_\Omega \). Then there exists some algebraic variety \( Y_0 \) over \( k \) such that \( Y = Y_0, \Omega \) by Lemma 4.17. It is clear that \( X(k) \subset Y_0(\Omega) \). Hence we have \( Y_0 = X \) since \( X(k) \) is Zariski dense in \( X \). Therefore we have \( Y = X_\Omega \).
Lemma 4.19. Let $\Omega/k$ be a field extension, $X_1$ an algebraic variety over $\Omega$ and $X_2$ an algebraic varieties over $k$. If $X_2(k)$ is Zariski dense in $X_2$, then $X_1(\Omega) \times X_2(k)$ is Zariski dense in $X_1 \times_\Omega X_2$. where $\Omega$ is an algebraic closure of $\Omega$.

Proof. Let $V$ be the Zariski closure of $X_1(\Omega) \times X_2(k)$ in $X_1 \times_\Omega X_2$. We assume that $V \subseteq X_1 \times_\Omega X_2$. Then we have an element $(x, y) \in (X_1(\Omega) \times X_2(\Omega)) \setminus V$. Therefore we have $(\{x\} \times X_2) \cap V \subseteq \{x\} \times X_2$ and $\{x\} \times X_2(k) \subset (\{x\} \times X_2(\Omega)) \cap V(\Omega)$. On the other hand, since $X_2(k)$ is Zariski dense in $X_2$, $X_2(k)$ is Zariski dense in $X_2$ by Corollary 4.18. Then $(x, y) \times X_2(k)$ is Zariski dense in $(x, y) \times_\Omega X_2$. This is a contradiction.

Theorem 4.20. Let $F'/F$ be an algebraic extension of fields such that $\Gamma(F')$ is Zariski dense in $\Gamma_F$. Assume that $F' = F$ or $\Lambda_1/F$ is a regular extension for all $l$. Then we have $(\Lambda')^{\Gamma(F')} = E'$ and $\Lambda \cap (\Lambda')^{\Gamma(F')} = E$.

Proof. The second part follows from the first part and the assumptions. Thus we prove the first part. In the proof of this theorem, we regard $l$ as an element of the index set $Z/d'$. We take any element $f = (f_l) \in (\Lambda')^{\Gamma(F')} \subset \prod_{l \in Z/d'} \Lambda_l'$, and consider $f_l$ as a rational function of $Z_l$ at $\Lambda_l$. Then, for some non-empty open affine set $U_l \subset Z_l$, $f_l$ can be regarded as a morphism $f_l : U_l \to \Lambda_l$. By Proposition 4.11, we have an isomorphism $Z_{E'} \times_{E'} \Gamma_{E'} \to Z_{E'} \times_{E'} \Gamma_{F', E}$; $(x, y) \mapsto (x, xy)$. We set $U \subset Z_{E'} \times_{E'} \Gamma_{E'}$ to be the open subset corresponding to $\mathbb{P}U_l \times_{E'} \mathbb{P}U_l$ via this isomorphism, and consider the two maps

$$g_i : U \longrightarrow \mathbb{P}U_l \times_{E'} \mathbb{P}U_l \stackrel{\pi_i}{\longrightarrow} \mathbb{P}U_l \stackrel{f=(f_l)}{\longrightarrow} \Lambda_l,$$

where $i = 1, 2$ and $\pi_i$ is the $i$-th projection. Let $S$ be an algebraic closure of $E'$. Then for any $(x, y) \in (Z(S) \times \Gamma(F')) \cap U(S)$, we have $g_1(x, y) = f(\pi_1(x, xy)) = f(x)$ and $g_2(x, y) = f(\pi_2(x, xy)) = f(xy) = f(x)$ since $f$ is fixed by $\Gamma(F')$. Since $\Gamma(F')$ is Zariski dense in $\Gamma_{E'}$, $Z(S) \times \Gamma(F')$ is Zariski dense in $Z_{E'} \times_{E'} \Gamma_{E'}$ by Lemma 4.19. Then $(Z(S) \times \Gamma(F')) \cap U(S)$ is Zariski dense in $U$. Thus we have $g_1 = g_2$, and this means $f_\pi_1 = f_\pi_2$. By considering on the level of coordinate rings, it is clear that $f \in E'$ since $E'$ is a field.

Corollary 4.21. If $F$ is a local field, and each connected component of $\Gamma$ has an $F$-valued point, then $\Lambda^{\Gamma}(F) = E$.

Proof. Take any connected component $\Gamma'$ of $\Gamma$. Then there exists an $F$-valued point $x \in \Gamma'(F)$ by the assumption, and $\Gamma'$ is smooth by Lemma 4.14. By the implicit function theorem, there exists an open neighborhood of $x$ in $\Gamma'(F)$ which is isomorphic to some open subset of $F^{\dim \Gamma}$. Since $\Gamma'$ is irreducible, this implies that $\Gamma'(F)$ is Zariski dense in $\Gamma'$. Hence we conclude that $\Gamma(F)$ is Zariski dense in $\Gamma$. Then this corollary follows from Theorem 4.20.

5 The group $\Gamma$ and $\varphi$-modules

5.1 General case

In this subsection, we use the notations defined in Section 3, and fix a $\sigma$-admissible triple $(F, E, L)$. Let $M \in \Phi M_E^L$ be an $L$-trivial $\varphi$-module over $E$ of rank $r$, $\mathcal{T}_M$ the Tannakian subcategory of $\Phi M_E^L$ generated by $M, V_M : \mathcal{T}_M \to \text{Vec}(F)$ the fiber functor of $\mathcal{T}_M$ and $\Gamma_M$ the Tannakian Galois group of $(\mathcal{T}_M, V_M)$. We fix $m \in \text{Mat}_{r \times 1}(M)$ an $E$-basis of $M$. Then
there exist matrices $\Phi \in \text{GL}_r(E)$ and $\Psi \in \text{GL}_r(L)$ such that $\varphi m = \Phi m$ and $\sigma \Psi = \Phi \Psi$. We define $\Gamma, \Sigma, \ldots$ as in Section 4 for $\Phi$ and $\Psi$. In this subsection, we show that there exists an equivalence of categories $T_M \xrightarrow{\sim} \text{Rep}(\Gamma, F)$ under some assumptions. Note that $\Sigma$ and $\Sigma_l$ are independent of the choice of $\mathfrak{m}$ and $\Psi$ by Proposition 3.13. If $N \in T_M$ and $s$ is the rank of $N$ over $E$, we use the notation $\mathfrak{n} \in \text{Mat}_{s \times 1}(N)$ and $\Psi_N \in \text{GL}_s(L)$ for an $E$-basis of $N$ and a fundamental matrix respectively. For any $F$-algebras $S$ and $R$, we set $S^{(R)} := R \otimes_F S$.

**Proposition 5.1.** For any $N \in T_M$, we have $\Psi_N \in \text{GL}_s(\Sigma)$.

**Proof.** Let $N$ and $N'$ be objects in $T_M$. Set $s := \dim_EN$ and $s' := \dim_E N$ and assume that $\Psi_N \in \text{GL}_s(\Sigma)$ and $\Psi_N' \in \text{GL}_{s'}(\Sigma)$. Since we can take $\Psi_{N \oplus N'} = \Psi_N \oplus \Psi_N'$, $\Psi_{N \otimes N'} = \Psi_N \otimes \Psi_N'$ and $\Psi_{N'} = (\Psi_N^{-1})^T$, we have that $\Psi_{N \oplus N'}$, $\Psi_N \otimes N'$ and $\Psi_N'$ are invertible matrices with coefficients in $\Sigma$. We have to show that if $0 \to N' \to N \to N'' \to 0$ is an exact sequence in $T_M$ and $\Psi_N \in \text{GL}_s(\Sigma)$, then $\Psi_N' \in \text{GL}_{s'}(\Sigma)$ and $\Psi_N'' \in \text{GL}_{s''}(\Sigma)$. Let $\mathfrak{n}, \mathfrak{n}'$ and $\mathfrak{n}''$ be $E$-bases of $N$, $N'$ and $N''$ such that

$$\mathfrak{n} = \begin{bmatrix} \mathfrak{n}' \\ \mathfrak{n}'' \end{bmatrix}$$

where $\mathfrak{n}''$ is a lift of $\mathfrak{n}''$. Since $V_M$ is exact, we have an exact sequence

$$0 \to V_M(N') \to V_M(N) \to V_M(N'') \to 0.$$

Let $x, x'$ and $x''$ be $F$-bases of $V_M(N), V_M(N')$ and $V_M(N'')$ such that

$$x = \begin{bmatrix} x' \\ x'' \end{bmatrix}$$

where $x''$ is a lift of $x''$. By Proposition 3.14, there exist matrices $A \in \text{GL}_s(F)$, $A' \in \text{GL}_{s'}(F)$ and $A'' \in \text{GL}_{s''}(F)$ such that $\Psi_N^{-1} \mathfrak{n} = Ax$, $\Psi_N^{-1} \mathfrak{n}' = A'x'$ and $\Psi_N^{-1} \mathfrak{n}'' = A''x''$. Consider the exact sequence

$$0 \to L \otimes_E N' \to L \otimes_E N \to L \otimes_E N'' \to 0.$$

Since both $\Psi_{N''}A''x''$ and $\mathfrak{n}''$ are mapped to $\mathfrak{n}''$ and $x'$ is an $L$-basis of $L \otimes_E N'$, there exists a matrix $B \in \text{Mat}_{s'' \times s'}(L)$ such that

$$\mathfrak{n}'' = Bx' + \Psi_{N''}A''x''.$$

Therefore we have

$$\Psi_N Ax = \mathfrak{n} = \begin{bmatrix} \mathfrak{n}' \\ \mathfrak{n}'' \end{bmatrix} = \begin{bmatrix} \Psi_N A' & 0 \\ B & \Psi_{N''}A'' \end{bmatrix} x.$$

Since $\Psi_N \in \text{GL}_s(\Sigma)$, we conclude that $\Psi_N' \in \text{GL}_{s'}(\Sigma)$ and $\Psi_N'' \in \text{GL}_{s''}(\Sigma)$.

**Lemma 5.2.** For any $N \in T_M$ and $F$-algebra $R$, there exists a natural isomorphism

$$\Sigma^{(R)} \otimes_F V(N) \to \Sigma^{(R)} \otimes_E N.$$

Similarly, there exists a natural isomorphism

$$\Sigma_l^{(R)} \otimes_F V(N) \to \Sigma_l^{(R)} \otimes_E N$$

for all $l$. 22
Proof. The inclusion $V(N) \subset \Sigma \otimes E N$ and the product map $\Sigma \otimes F \Sigma \rightarrow \Sigma$ induce a natural map

$$\kappa : \Sigma^{(R)} \otimes F V(N) \hookrightarrow \Sigma^{(R)} \otimes F \Sigma \otimes E N \rightarrow \Sigma^{(R)} \otimes E N.$$ 

Since $1 \otimes \Psi^{-1}_N n$ is a $\Sigma^{(R)}$-basis of $\Sigma^{(R)} \otimes F V(N)$, we can write $\kappa$ explicitly as follows:

$$\kappa(f \cdot (1 \otimes \Psi^{-1}_N n)) = (f \Psi^{-1}_N) \cdot (1 \otimes n)$$

for all $f \in \text{Mat}_{1 \times 1}(\Sigma^{(R)})$. Hence it is clear that $\kappa$ is an isomorphism. The $\Sigma_l$ version is proved by the same argument. \qed

**Theorem 5.3.** For any $N \in \mathcal{T}_M$, there exists a natural representation

$$\rho_N : \Gamma \rightarrow \text{GL}(V(N))$$

over $F$ that is functorial in $N$.

**Proof.** For any $F$-algebra $R$ and $\gamma \in \Gamma(R) \subset \text{GL}_r(R)$, we define

$$\rho^{(R)}_N(\gamma) : R \otimes F V(N) \hookrightarrow \Sigma^{(R)} \otimes F V(N) \rightarrow \Sigma^{(R)} \otimes E N \rightarrow \Sigma^{(R)} \otimes E N,$$

where the second map is the isomorphism defined in Lemma 5.2 and the third map is defined by $h(\Psi) \otimes x \mapsto h(\Psi \gamma) \otimes x$. Clearly $\rho^{(R)}_N$ is functorial in $N$. If $\text{im}(\rho^{(R)}_N(\gamma)) = R \otimes F V(M)$ then $\text{im}(\rho^{(R)}_N(\gamma)) = R \otimes F V(N)$ for all $N \in \mathcal{T}_M$. Thus we may assume that $N = M$. We can write $\rho^{(R)}_M(\gamma)$ explicitly:

$$\rho^{(R)}_M(\gamma)(f \cdot (1 \otimes \Psi^{-1} m)) = f \gamma^{-1}(1 \otimes \Psi^{-1} m),$$

for each $f \in \text{Mat}_{1 \times r}(R)$. Therefore we have $\text{im}(\rho^{(R)}_M(\gamma)) = R \otimes F V(M)$.

From the above description of $\rho^{(R)}_M$, we have the following corollary:

**Corollary 5.4.** The representation $\rho_M : \Gamma \rightarrow \text{GL}(V(M))$ is faithful.

From Theorem 5.3, we have a functor $\xi_M : \mathcal{T}_M \rightarrow \text{Rep}(\Gamma, F)$, and it is clear by the construction that $\xi_M$ is a tensor functor. Let $\eta_M : \text{Rep}(\Gamma_M, F) \rightarrow \mathcal{T}_M$ be the equivalence of categories defined by the Tannakian duality and $\alpha : \text{Rep}(\Gamma, F) \rightarrow \text{Vec}(F)$ the forgetful functor. Since $V_M = \alpha \circ \xi_M$, there exists a unique homomorphism $\pi_M : \Gamma \rightarrow \Gamma_M$ over $F$ such that the natural functor $\tau_M : \text{Rep}(\Gamma_M, F) \rightarrow \text{Rep}(\Gamma, F)$ induced by $\pi_M$ satisfies $\xi_M \circ \eta_M = \tau_M$.

$$\text{Rep}(\Gamma_M, F) \xrightarrow{\eta_M} \mathcal{T}_M \xrightarrow{\xi_M} \text{Rep}(\Gamma, F) \xrightarrow{\alpha} \text{Vec}(F)$$

**Proposition 5.5.** For any representation $W \in \text{Rep}(\Gamma, F)$, there exists an object $N \in \mathcal{T}_M$ such that $W$ is isomorphic to a subquotient of $\xi_M(N)$. 

23
Proof. By Corollary 5.4, the $\Gamma$-representation $\xi_M(M) = \rho_M$ is faithful. Therefore, $W$ is isomorphic to a subquotient of representation of the form

$$\oplus_{i=1}^n (\xi_M(M))^\otimes a_i \otimes (\xi_M(M)^\vee)^\otimes b_i,$$

where $a_i, b_i \in \mathbb{N}$. However we have $\oplus_{i=1}^n (\xi_M(M))^\otimes a_i \otimes (\xi_M(M)^\vee)^\otimes b_i = \xi_M(\oplus_{i=1}^n M^\otimes a_i \otimes (M^\vee)^\otimes b_i).$

Proposition 5.5 is equivalent to the next theorem ([6], Proposition 2.21).

**Theorem 5.6.** The morphism of affine $F$-schemes $\pi_M : \Gamma \rightarrow \Gamma_M$ is a closed immersion.

From now on, we assume that $\Gamma(F)$ is Zariski dense in $\Gamma$ or $\Lambda_1/F$ is a regular extension for each $l$. In the former case we put $F' = F$, and in the latter case we put $F' = \bar{F}$. For any $F$-algebra $S$, we set $S' := F' \otimes_F S$. Then in any case, $E'$ and $\Lambda'_l$ are fields, $\Lambda'_l = \text{Frac}(\Sigma'_l)$ and $\Lambda \cap (\Lambda')^1(F') = E$ by Theorem 4.20.

**Proposition 5.7.** Assume that $\Gamma(F)$ is Zariski dense in $\Gamma$ or $\Lambda_1/F$ is a regular extension for each $l$. Then the functor $\xi_M : T_M \rightarrow \text{Rep}(\Gamma, F)$ is fully faithful.

Proof. For any objects $N, N' \in T_M$, there exist natural isomorphisms $\text{Hom}_{T_M}(N', N) \cong \text{Hom}_{T_M}(1, \text{Hom}(N', N))$ and $\text{Hom}_{T_M}(V(N'), V(N)) \cong \text{Hom}_{T_M}(V(1), V(\text{Hom}(N', N)))$. Thus it is enough to show that, for any $N \in T_M$, $\text{Hom}_{T_M}(1, N) \rightarrow \text{Hom}_{T_M}(V(1), V(N))$ is an isomorphism. It is injective since $\text{Hom}_{T_M}(1, N) = N^\vee = N \cap V(N) \rightarrow \text{Hom}_{T_M}(V(1), V(N))$.

For any $\phi \in \text{Hom}_{T_M}(V(1), V(N))$, there exists $h = h(\Psi) \in \text{Mat}_{1 \times s}(\Sigma)$ so that $\phi(1) = \text{hn}$ by Lemma 5.2. Then for any $\gamma \in \Gamma(F')$, we have $h(\Psi \gamma) = h(\Psi \gamma) = \gamma h(\Psi \gamma)$.

Hence $h(\Psi) = h(\Psi) = \gamma h$. By Theorem 4.20, we have $h \in \text{Mat}_{1 \times s}(E)$, and this implies $\phi(1) = \text{hn} \in N \cap V(N)$.

We prepare a lemma from linear algebra.

**Lemma 5.8.** Let $E \subset \Lambda$ be general rings where $E$ is a field and $\Lambda = \prod_{l \in \mathbb{Z}/d'} \Lambda_l$ is a finite product of fields. Assume that $\#E > d'$. Let $1 \leq m \leq s$ and $D \in \text{Mat}_{s \times m}(\Lambda)$. If there exists $D_0 \in \text{GL}_d(\Lambda)$ such that $D_0 = [*, D]$, then there exist $A \in \text{GL}_m(\Lambda)$ and $B \in \text{GL}_s(E)$ such that

$$BDA = \begin{bmatrix} 1 & \cdots & 1 \\ * & * & * \end{bmatrix} \in \text{Mat}_{s \times m}(\Lambda).$$

Proof. For each $l \in \mathbb{Z}/d'$ and $1 \leq j \leq m$, let $e_{l,j} \in \text{Mat}_{1 \times m}(\Lambda_l)$ be a row vector such that the $j$-th component is one and the other components are zero. Write $D = (D_l)_l$ where $D_l \in \text{Mat}_{s \times m}(\Lambda_l)$. Since the rank of $D_l$ is $m$ for each $l$, there exists a matrix $A_l \in \text{GL}_m(\Lambda_l)$ such that

$$D_l A_l = \begin{bmatrix} C_{l,1} \\ \vdots \\ C_{l,s} \end{bmatrix},$$

where $C_{l,j} \in \text{Mat}_{1 \times m}(\Lambda_l)$, and for each $1 \leq j \leq m$ there exists an $i$ such that $C_{l,i} = e_{l,j}$. An elementary pattern of $D_l A_l$ is a choice of $(i_1, \ldots, i_m) \in \{1, \ldots, s\}^m$ such that $C_{l,ik} = e_{l,k}$ for each $1 \leq k \leq m$. We fix an elementary pattern $(i_{l1}, \ldots, i_{lm})$ of $D_l A_l$ for each $l$. For
each matrix $P \in \text{Mat}_{s \times m}(\Lambda_l)$ such that, for each $1 \leq j \leq m$ there exists an $i$ such that the $i$-th row of $P$ is $e_{l,j}$, we define an elementary pattern of $P$ in the same way. For a matrix in $\text{Mat}_{s \times m}(\Lambda)$, we define the procedures

(1) left-multiplication by a matrix in $\text{GL}_s(E)$,

(2) right-multiplication by a matrix in $\text{GL}_m(\Lambda_l) \times \prod_{l' \neq l}\{1\}$.

Set $\tilde{A} := (\tilde{A}_l)_l \in \text{GL}_m(\Lambda)$ and $C_l := (C_{l,i})_l \in \text{Mat}_{1 \times m}(\Lambda_l)$. By using the above procedures, we want to transform $\tilde{D}$ to a matrix $D' = (D_l')_l$ such that, we can choose an elementary pattern of $D_l'$ to $(1, \ldots, m)$ for each $l$.

Fix $i' \neq i''$ and $l_0$. Let $\tau = (i' i'')$ be the transposition of $i'$ and $i''$. It is enough to show that, by using the procedures (1) and (2l), we can transform $\tilde{D}$ to a matrix $D' = (D_l')_l$ such that, we can choose an elementary pattern of $D_{l_0}'$ to $(\tau i_{l_0,1}, \ldots, \tau i_{l_0,m})$ and an elementary pattern of $D_l'$ to $(i_{l,1}, \ldots, i_{l,m})$ for each $l \neq l_0$.

First we assume that $i' = i_{l_0,j'}$ and $i'' = i_{l_0,j''}$ for some $j' \neq j''$. For $c \in E^s$, we can exchange the $i'$-th row of $\tilde{D}$ for $C_{l'} + cC_{l''}$ by the procedure (1). Since $\#E > d'$, we can take $c$ such that, for each $l \neq l_0$, if $i' = i_{l,j}$ for some $j$ then the $j$-th component of $C_{l'} + cC_{l''}$ is non-zero. Then by the procedures (2) for $l \neq l_0$, we can transform this matrix to a matrix $D'' = (D''_l')_l$ such that, we can choose an elementary pattern of $D''_{l_0}$ to $(i_{l_1,1}, \ldots, i_{l_m})$ for each $l \neq l_0$, the $i$-th row of $D''_{l_0}$ is $C_{l_0,i}$ for each $i \neq i'$ and the $i'$-th row of $D''_{l_0}$ is $(\tilde{j'}, \tilde{j''}, 0, \ldots, 0, 0, \tilde{c}, 0, \ldots, 0)$. Therefore by the procedure (2l), we can transform $D''$ to a matrix $D'$ which has the desired properties. The case that $i' \not\in (i_{l_0,1}, \ldots, i_{l_0,m})$ and $i'' = i_{l_0,j''}$ for some $j''$ is proved in a similar way, and we omit the proof.

Lemma 5.9. Assume that $\Gamma(F)$ is Zariski dense in $\Gamma$ or $\Lambda_l/F$ is a regular extension for each $l$. Assume also that $\#E > d'$. We take $1 \leq m \leq s$ and $D \in \text{Mat}_{s \times m}(\Lambda)$ such that $[*, D] \in \text{GL}_s(\Lambda)$ for some $* \in \text{Mat}_{s \times (s-m)}(\Lambda)$. We set

$$W := \{x \in \text{Mat}_{1 \times s}(\Lambda^\prime)|x D = 0\},$$

and assume that $\Gamma(F^\prime)W \subset W$, where the elements of $\Gamma(F^\prime)$ act on $W$ by componentwise. Then there exists a matrix $C \in \text{Mat}_{(s-m) \times s}(E)$ such that the rank of $C$ is $s - m$ and $CD = 0$.

Proof. By Lemma 5.8, there exist matrices $A \in \text{GL}_m(\Lambda)$ and $B \in \text{GL}_s(E)$ such that

$$BDA = \begin{bmatrix} I_m \\ C_0 \end{bmatrix},$$

where $I_m$ is the identity matrix of size $m$ and $C_0 \in \text{Mat}_{(s-m) \times m}(\Lambda)$. We set

$$W_B := WB^{-1} = \{x \in \text{Mat}_{1 \times s}(\Lambda^\prime)|x B D = 0\} = \{x \in \text{Mat}_{1 \times s}(\Lambda^\prime)|x \begin{bmatrix} I_m \\ C_0 \end{bmatrix} = 0\}.$$

Then it is clear that $W_B$ is also $\Gamma(F^\prime)$-stable. Thus, since each row of $[-C_0 I_{s-m}]$ is an element of $W_B$, each row of $[-\gamma C_0 I_{s-m}]$ is also an element of $W_B$ for any $\gamma \in \Gamma(F^\prime)$. 25
This means that $\gamma C_0 = C_0$ for each $\gamma \in \Gamma(F')$. Therefore $C_0 \in \text{Mat}(s_{-m} \times m(E))$ by Theorem 4.20. We set $C := [-C_0 \quad I_{s-m}] \mathbf{B}$. Then it is clear that this $C$ has the desired properties. 

\begin{proposition}
Assume that $\Gamma(F)$ is Zariski dense in $\Gamma$ or $\Lambda_l/F$ is a regular extension for each $l$. Assume also that $\# E > d'$. For any $N \in \mathcal{T}_M$ and $\Gamma$-subrepresentation $U \subset \xi_M(N)$, there exists a $\varphi$-submodule $N' \subset N$ such that $\xi_M(N') = N$.
\end{proposition}

\begin{proof}
We take $u \in \text{Mat}_{u \times 1}(U)$ an $F$-basis of $U$ such that $\mathbf{n} := [u \quad \ast]^\text{tr}$ forms an $F$-basis of $\xi_M(N)$. By Lemma 5.2, we have $\mathbf{n} = H \mathbf{n}$ for some $H = H(\Psi) \in \text{GL}_s(\Sigma)$. We take a matrix $D \in \text{Mat}_{s \times (s-u)}(\Sigma)$ such that $H^{-1} = [\ast \quad D]$, and set $W := \{x \in \text{Mat}_{1 \times s}(\Lambda') | xD = 0\}$. Since $I_s = HHH^{-1} = [\ast \quad HD]$, the $i$-th row of $H$ is an element of $W$ for each $i \leq u$. These form a $\Lambda'$-basis of $W$ because the coefficient ring $\Lambda'$ is a finite product of fields. For each $\gamma \in \Gamma(F')$, we have $\gamma \mathbf{n} = (\gamma H) \mathbf{n} = (\gamma H)H^{-1} \mathbf{n}$. Since $U$ is $\Gamma$-stable, the $(i, j)$-th component of $(\gamma H)H^{-1} = [\ast \quad (\gamma H)D]$ is zero for each $i \leq u$ and $j > u$. Therefore, $W$ is $\Gamma(F')$-stable. By Lemma 5.9, there exists a matrix $C \in \text{Mat}_{u \times s}(E)$ such that the rank of $C$ is $u$ and $CD = 0$. Then we can take $B \in \text{GL}_s(E)$ such that $C$ forms the top rows of $B$. Let $[\mathbf{n}' \quad \mathbf{n}'']^\text{tr} := B \mathbf{n}$ where $\mathbf{n}' \in \text{Mat}_{u \times 1}(N)$. Let

$$BH^{-1} = \begin{bmatrix} C' & \ast \\ \ast & D \end{bmatrix} = \begin{bmatrix} \Psi' & 0 \\ \ast & \ast \end{bmatrix},$$

where $\Psi' \in \text{GL}_u(\Sigma)$. Then we have

$$\varphi \begin{bmatrix} \mathbf{n}' \\ \mathbf{n}'' \end{bmatrix} = \varphi(B \mathbf{n}) = \varphi(BH^{-1} \mathbf{n}) = \sigma(BH^{-1})\varphi(H \mathbf{n}) = \sigma(BH^{-1})H \mathbf{n}$$

$$= \sigma(BH^{-1})HB^{-1} \begin{bmatrix} \mathbf{n}' \\ \mathbf{n}'' \end{bmatrix} = \begin{bmatrix} \Psi' & 0 \\ \ast & \ast \end{bmatrix} \begin{bmatrix} \mathbf{n}' \\ \mathbf{n}'' \end{bmatrix},$$

where $\Psi' \in \text{GL}_u(E)$. Hence $N' := (\mathbf{n}')_E \subset N$ is a sub $\varphi$-module, and we have $\varphi \mathbf{n}' = \Phi' \mathbf{n}'$. Moreover, we have

$$\sigma \begin{bmatrix} \Psi' & 0 \\ \ast & \ast \end{bmatrix} = \sigma(BH^{-1}) = (\sigma(BH^{-1})HB^{-1})(BH^{-1}) = \begin{bmatrix} \Psi' & 0 \\ \ast & \ast \end{bmatrix} \begin{bmatrix} \Psi' & 0 \\ \ast & \ast \end{bmatrix} = \begin{bmatrix} \Psi' \Psi' & 0 \\ \ast & \ast \end{bmatrix}.$$ 

Therefore, $\Psi'$ is a fundamental matrix for $\Phi'$. Since

$$\begin{bmatrix} \mathbf{n}' \\ \mathbf{n}'' \end{bmatrix} = B \mathbf{n} = BH^{-1} \mathbf{n} = \begin{bmatrix} \Psi' & 0 \\ \ast & \ast \end{bmatrix} \begin{bmatrix} \mathbf{n}' \\ \mathbf{n}'' \end{bmatrix},$$

we have that $\xi_M(N') = (\Psi')^{-1} \mathbf{n}' = (\mathbf{u})_F = U$. 

\begin{theorem}
Assume that $\Gamma(F)$ is Zariski dense in $\Gamma$ or $\Lambda_l/F$ is a regular extension for each $l$. Assume also that $\# E > d'$. Then the morphism of affine $F$-schemes $\pi_M : \Gamma \rightarrow \Gamma_M$ is an isomorphism. Equivalently, the functor $\xi_M : \mathcal{T}_M \rightarrow \text{Rep}(\Gamma, F)$ is an equivalence of Tannakian categories.
\end{theorem}

\begin{proof}
By Propositions 5.7 and 5.10, $\pi_M$ is faithfully flat ([6], Proposition 2.21). On the other hand, $\pi_M$ is a closed immersion by Theorem 5.6. Therefore $\pi_M$ is an isomorphism.
\end{proof}
5.2 \( v \)-adic case

In this subsection, we continue to use the notations of the previous subsection and consider the case that \((F, E, L) = (\mathbb{F}_q(t)_v, K(t)_v, K^{sep}(t)_v)\), where the notations are defined in Subsection 3.3.

The assumption that \( A_l/F \) is regular for each \( l \) is not true in general. For example, assume that \( r = 1, v = t \) and \( \Phi \in K \) such that \( \Psi := \Phi^{1/(q-1)} \notin K(t)_v \). Then \( \Psi \) is a fundamental matrix for \( \Phi \), and it is clear that \( Z \) is not absolutely irreducible. Therefore the assumptions are not satisfied. However we expect that this assumption is true for “good” objects.

Hence we consider the other assumption. In the \( v \)-adic case, \( \Gamma(\mathbb{F}_q(t)_v) \) contains a Galois image. Since the Galois image is large enough, we can conclude that \( \Gamma(\mathbb{F}_q(t)_v) \) is Zariski dense in \( \Gamma \).

**Lemma 5.12.** Let \( G \) be an algebraic group over a field \( k \) and \( H \) a subgroup of \( G(k) \). We set \( H^{Zar} \) the Zariski closure of \( H \) in \( G \) endowed with the reduced structure. Then \( H^{Zar} \) is a subgroup scheme of \( G \) and smooth.

**Proof.** We denote by \( \bar{H} \) the Zariski closure of \( H \) in \( G(\bar{k}) \). By Lemma 4.17, \( \bar{H} \) is defined over \( k \). Then it is clear that \( (H^{Zar})_k = \bar{H} \). Thus \( H^{Zar} \) is absolutely reduced.

To prove that \( H^{Zar} \) is a group scheme, it is enough to show that \( \bar{H} \) is a group. For any \( a \in G(\bar{k}) \), the map \( G(\bar{k}) \to G(\bar{k}); g \mapsto ag \) is a homeomorphism. Thus for any \( a \in H \), we have \( a\bar{H} = \bar{aH} \subset \bar{H} \). Thus for any \( b \in \bar{H} \), we have \( Hb \subset \bar{H} \). Therefore \( Hb = \bar{H}b \subset \bar{H} \).

Hence we have \( \bar{H}H \subset \bar{H} \). Since the map \( G(\bar{k}) \to G(\bar{k}); g \mapsto g^{-1} \) is a homeomorphism, we have \( \bar{H}^{-1} = \overline{H^{-1}} = \bar{H} \).

**Lemma 5.13.** Let \( G \) be a topological group and \( k \) be a topological field. Let \( \rho : G \to \text{GL}_r(k) \) be a continuous \( k \)-representation of \( G \). We set \( C_\rho \) the Tannakian subcategory of \( \text{Rep}(G, k) \) generated by \( \rho \) and \( \Gamma(\rho) \subset \text{GL}_r/k \) its Tannakian Galois group. Then \( \rho(G) \) is Zariski dense in \( \Gamma(\rho) \).

Note that the Tannakian Galois group \( \Gamma(\rho) \) may not be reduced.

**Proof.** We have an inclusion \( \rho(G)^{Zar} \subset \Gamma(\rho) \) and \( \rho \) factors through a \( \rho(G)^{Zar}(k) \):

\[ \rho : G \to \rho(G)^{Zar}(k) \twoheadrightarrow \Gamma(\rho) \leftarrow \rho(G) \twoheadrightarrow \text{GL}_r(k). \]

Thus we have functors of Tannakian categories

\[ C_\rho \cong \text{Rep}(\Gamma(\rho), k) \to \text{Rep}(\rho(G)^{Zar}, k) \to \text{Rep}(G, k). \]

We denote by \( \Gamma_{G,k} \) the Tannakian Galois group of \( \text{Rep}(G, k) \). Then we have morphisms of algebraic groups which correspond to the above sequence:

\[ \Gamma_{G,k} \to \rho(G)^{Zar} \twoheadrightarrow \Gamma(\rho). \]

Since \( \Gamma_{G,k} \to \Gamma(\rho) \) is an epimorphism of algebraic groups, we have \( \rho(G)^{Zar}(k) = \Gamma(\rho)(k) \).

For any \( \tau \in G_K \), since \( \sigma(\tau \Psi) = \tau(\sigma \Psi) = \tau(\Phi \Psi) = \Phi(\tau \Psi) \), there exists a matrix \( A_\tau \in \text{GL}_r(\mathbb{F}_q(t)_v) \) such that \( \tau \Psi = \Psi A_\tau \). Therefore we have \( \tau(\Sigma) = \Sigma \) and a map \( G_K \to \text{Aut}_\sigma(\Sigma/K(t)_v) \). By Lemma 4.16, we have that \( A_\tau \in \Gamma(\mathbb{F}_q(t)_v) \) and \( A_\tau \) corresponds to the image of \( \tau \) in \( \text{Aut}_\sigma(\Sigma/K(t)_v) \) via the isomorphism \( \text{Aut}_\sigma(\Sigma/K(t)_v) \cong \Gamma(\mathbb{F}_q(t)_v) \).
On the other hand, we can verify that the map

\[ G_K \to \text{Aut}_\sigma(\Sigma/K(t)_v) \cong \Gamma(F_q(t)_v) \to \Gamma_M(F_q(t)_v) \to \text{GL}(V(M)) \]

coincide with the natural representation \( G_K \to \text{GL}(V(M)) \) defined in Subsection 3.3.

**Proposition 5.14.** The image of \( G_K \) in \( \Gamma_M(F_q(t)_v) \) is Zariski dense in \( \Gamma_M \).

**Proof.** Let \( C_M \) be the Tannakian subcategory of \( \text{Rep}(G_K, F_q(t)_v) \) generated by \( V(M) \). Then by Theorem 3.26, the categories \( T_M \) and \( C_M \) are equivalence. Therefore \( \Gamma_M \) is also a Tannakian Galois group of \( C_M \). Hence by Lemma 5.13, the image of \( G_K \) is Zariski dense in \( \Gamma_M \).

**Theorem 5.15.** If \((F, E, L) = (F_q(t)_v, K(t)_v, K^{\text{sep}}(t)_v)\), then the morphism \( \pi_M : \Gamma \to \Gamma_M \) is an isomorphism.

**Proof.** By Proposition 5.14, \( \Gamma(F_q(t)_v) \) is Zariski dense in \( \Gamma_M \). In particular, it is Zariski dense in \( \Gamma \). Therefore by Theorem 5.11, \( \pi_M \) is an isomorphism.

**Proposition 5.16.** Fix an index \( m \in \mathbb{Z}/d \) and take an element \( \tau \in G_K \) such that \( \tau|_{F_q} = \sigma|_{F_q}^{-m} \). Then the image of \( \tau \) in \( \Gamma(F_q(t)_v) \) is contained in \( \Gamma_m(F_q(t)_v) \).

**Proof.** Since \( \tau \) induces a \( K(t)_v \)-isomorphism \( K^{\text{sep}}((t - \lambda_{l+m})) \to K^{\text{sep}}((t - \lambda_l)) \), also induces a bijection \( Z_{l+m}(K^{\text{sep}}((t - \lambda_{l+m}))) \to Z_{l+m}(K^{\text{sep}}((t - \lambda_l))) \). Let \( A_\tau \in \Gamma(F_q(t)_v) \) be as above. Since \( \Psi A_\tau = \tau \Psi \), we have \( \Psi A_\tau = \tau \Psi |_{Z_{l+m}(K^{\text{sep}}((t - \lambda_l)))} \). Note that \( \Psi |_{Z_{l+m}(K^{\text{sep}}((t - \lambda_l)))} \) for each \( l \) by the definition of \( \Psi |_{Z_{l+m}(K^{\text{sep}}((t - \lambda_l)))} \). Therefore by Theorem 4.11, we have \( A_\tau \in \Gamma_m(F_q(t)_v) \cap \Gamma(F_q(t)_v) = \Gamma_m(F_q(t)_v) \).

### 6 \( v \)-adic criterion

In this section, we set \( K := F_q(\theta) \) the rational function field over \( F_q \) with one variable \( \theta \) independent of \( t \). Let \( M \) be a finite-dimensional \( \varphi \)-module over \( K(t)_v \), \( \mathbf{m} \) a \( K(t)_v \)-basis of \( M \) and \( \Phi \in \text{Mat}_{r \times r}(K(t)_v) \) a matrix such that \( \varphi \mathbf{m} = \Phi \mathbf{m} \).

**Definition 6.1.** A \( \varphi \)-module \( M \) is said to be a \( v \)-adic \( t \)-motive if \( \Phi \in \text{Mat}_{r \times r}(K[t]_v) \) and \( \det \Phi = c(t - \theta)^s \) for some \( c \in \mathbb{K}^\times \) and \( s \in \mathbb{N} \).

Since \( t - \theta \) is invertible in \( K[t]_v \), \( v \)-adic \( t \)-motives are \( K^{\text{sep}}(t)_v \)-trivial by Theorem 3.26. Thus we can apply the results of the previous sections to \( v \)-adic \( t \)-motives.

**Remark 6.2.** Let \( k \) be a field of characteristic \( p > 0 \) and \( \iota : F_p[t] \to k \) a ring homomorphism. Anderson defined the notion of \( t \)-motives over \( k \) in [1]. This is a \( \varphi \)-module \( M \) over \( k[t] \) which satisfies the following conditions:

- \( M \) is free of finite rank over \( k[t] \).
- \( (t - \iota(t))^N(M/k[t] : \varphi M) = 0 \) for some integer \( N > 0 \).
- \( M \) is finitely generated over \( k_\sigma[\varphi] \).

Here the \( \varphi \) action on \( k[t] \) is defined as before and \( k_\sigma[\varphi] \) is the subring of \( k[t]_\sigma[\varphi] \) generated by \( k \) and \( \varphi \). Thus we have a functor from the category of \( t \)-motives over \( K \) (here we take \( \iota(t) = \theta \)) to the category of \( v \)-adic \( t \)-motives by tensoring \( K(t)_v \).
Let $K_{v(\theta)}$ be the completion of $K$ with respect to the place at $v(\theta)$, $K_d := K \cdot F_q^d$ the composite field of $K$ and $F_q^d$ in $\overline{K}$, $K_{\lambda_l} := K_{d,(\theta-\lambda_l)}$ the completion of $K_d$ with respect to the place at $(\theta-\lambda_l)$, $\overline{K_{\lambda_l}}$ an algebraic closure of $K_{\lambda_l}$, and $C_{\lambda_l} := \overline{K_{\lambda_l}}$ the completion of $\overline{K_{\lambda_l}}$ with respect to the canonical extension of $(\theta-\lambda_l)$. Let $v_l$ be the valuation on $C_{\lambda_l}$ normalized by $v_l(\theta-\lambda_l) = 1$. For each $l$, we fix an embedding $\overline{K}$ to $\overline{K_{\lambda_l}}$ over $K_d$. Then for each $f \in K_{v(\theta)}$, we can define $f(\theta) \in \prod_l C_{\lambda_l}$ by substituting $\theta$ for $t$ if it converge. We have the following conjecture, which is a $v$-adic analogue of Proposition 3.1.1 in [2]:

**Conjecture 6.3.** Let $\Phi \in GL_r(K(t)_v) \cap \text{Mat}_{r \times r}(K[t])$ and $\psi \in \text{Mat}_{r \times 1}(K_{v(\theta)}^\text{sep})$ be matrices such that $\psi(\theta)$ converges, $\sigma \psi = \Phi \psi$ and $\det \Phi = c(t-\theta)^s$ for some $c \in K^\times$ and $s \in \mathbb{N}$. Then, any linear relation of the components of $\psi(\theta)$ over $K_{v(\theta)}$ lifts to some linear relation of the components of $\psi$ over $K[t]_v$. Precisely speaking, if there exists an element $\rho \in \text{Mat}_{1 \times r}(K_{v(\theta)})$ such that $\rho \psi(\theta) = 0$, then there exists an element $P \in \text{Mat}_{1 \times r}(K[t]_v)$ such that $P \rho = 0$, $P(\theta)$ converges and $P(\theta) = \rho$.

Conjecture 6.3 is true if $r = 1$ and we give a proof below. This proof is the same as the proof of the $\infty$-adic version for $r = 1$ in [2].

If $\rho = 0$, then we can take $P = 0$. Therefore we may assume that $\rho \neq 0$. Since for some $P \in K[t]_v$, we have $P(\theta) = \rho$. Hence it is enough to show that, if $\psi(\theta) = 0$, then $\psi = 0$. Write $\psi = (\sum_i a_{l,i}(t-\lambda_l)^i)_{l,i}$. For any $\nu \geq 0$, the infinite sum $\sum_i a_{l,i}(\theta-\lambda_l)^i$ converges because $\sum_i a_{l,i}(\theta-\lambda_l)^i$ converges and $v_l((\theta-\lambda_l)^{q^{d \nu}}) \geq v_l(\theta-\lambda_l)$ for each $l$. Thus we have

$$\psi(\theta^{q^{d \nu}})^d = (\sum_i a_{l,i}(\theta-\lambda_l)^{q^{d \nu}})^i = (\sum_i a_{l,i}(\theta^{q^{d \nu}+1} - \lambda_l)^i = (\sigma^d \psi)(\theta^{q^{d \nu+1}}).$$

On the other hand, we have

$$(\sigma^d \psi)(\theta^{q^{d \nu+1}}) = (\sigma^{d-1} \Phi)(\theta^{q^{d \nu+1}}) \times \ldots \times (\sigma^0 \Phi)(\theta^{q^{d \nu+1}}) \times \psi(\theta^{q^{d \nu+1}})$$

$$= c^{q^{d-1}} \ldots q^{q^{d \nu}}(\theta^{q^{d \nu+1}} - \theta^{q^d \nu}) \ldots (\theta^{q^{d \nu+1}} - \theta^q \psi(\theta^{q^{d \nu+1}}).$$

By induction on $\nu$, we have $\sum_i a_{l,i}(\theta-\lambda_l)^{q^{d \nu}} = 0$ for each $l$. Thus the formal series $\sum_i a_{l,i}z^i$ has infinite zeros on the disk $v_l(z) \geq v_l(\theta-\lambda_l)$. Therefore $a_{l,i} = 0$ for all $l$ and $i$, and we conclude that $\psi = 0$.

Next, we calculate valuations of the coefficients of periods for some examples of $t$-motives. An element $L_{\alpha,i}$ is an analogue of the $n$-th Carlitz polylogarithm, and an element $\Omega_v$ is an analogue of the Carlitz period.

**Proposition 6.4.** Let $n \geq 1$ be an integer and $\alpha \in (K_{v(\theta)})^\times$ an element such that $v_l(\alpha) \geq 0$ for all $l$. Then there exists an element $L_{\alpha,i} = L_{\alpha,i}(t) = (\sum_i a_{l,i}(t-\lambda_l))_{l} \in K_{v(\theta)}^\text{sep} = \prod_l K_{v(\theta)}^\text{sep}[t-\lambda_l] \text{ which satisfies the equation}$

$$\sigma(L_{\alpha,i}) = \sigma(\alpha) + \alpha/(t-\theta)^n.$$  

For any $l \in \mathbb{Z}/d$, $0 \leq m \leq d-1$ and $i \geq 0$, we have

$$v_l(a_{l,m,i}) \geq -q^{n \frac{i}{q^d} + \frac{n}{q^d-1}}.$$
Proof. For an element \( L_{α,n} = (\sum a_{l,i}(t - λ_i))_l ∈ \prod K^{sep}(t - λ_i) \), we have an explicit descriptions
\[
(t - θ)^n σ(L_{α,n}) = \left( \sum_i \left( \sum_{j=0}^{n} \binom{n}{j} (λ_i - θ)^{n-j} a^{q}_{l-i,j} \right) (t - λ_i)^j \right)_l,
\]
\[
σ(α)(t - θ)^n = \left( \sum_{i=0}^{n} \binom{n}{i} (λ_i - θ)^{n-i} α^q(t - λ_i)^i \right)_l.
\]
We set \( b_{i,i} := \sum_{j=1}^{n} \binom{n}{j} (λ_i - θ)^{n-j} a^{q}_{l-i,j} - \binom{n}{j} (λ_i - θ)^{n-i} α^q \) and \( c_i := (λ_i - θ)^n \). Then the equation in Proposition is equivalent to the equations
\[
a_{l+1,i} = c_{l+1}a^q_{l,i} + b_{l+1,i}
\]
for all \( l ∈ \mathbb{Z}/d \) and \( i ∈ \mathbb{Z} \). For \( i < 0 \), we can take \( a_{l,i} = 0 \). Fix \( i ≥ 0 \) and consider the system of polynomial equations
\[
X_{l+1} = c_{l+1}X^q_l + b_{l+1,i} \quad (l ∈ \mathbb{Z}/d).
\]
For \( 2 ≤ r ≤ m \), we set
\[
β_{m,r,i} := b^q_{r,i} \prod_{s=r+1}^{m} c^q_{s} \quad and \quad γ_m := \prod_{s=2}^{m} c^q_{s}.
\]
Then the above equations are equivalent to the equations
\[
X_m = γ_mX^{q^{m-1}}_1 + \sum_{r=2}^{m} β_{m,r,i} \quad (2 ≤ m ≤ d + 1).
\]
Since \( X_{d+1} = X_1 \), we can solve these equations in \( K^{sep} \). This proved the existence part of this proposition.

Next we calculate the valuations of these solutions by induction on \( i \). We set \( f_i(X_1) := γ_{d+1}X^{q^d}_1 - X_1 + \sum_{r=2}^{d+1} β_{d+1,r,i} \). Since \( a_{i,i} = 0 \) for all \( i < 0 \), the inequality for the valuations in the statement of this proposition is true for \( i < 0 \). Fix \( i ≥ 0 \) and assume that the inequality in the statement of this proposition is true for integers lower than \( i \). It is clear that \( v_1(γ_{d+1}) = v_1(c_1) = n \). For \( 2 ≤ r ≤ d \), we have
\[
v_1(β_{d+1,r,i}) = n + q^{d+1-r}v_1(b_{r,i})
\]
\[
≥ n + q^{d+1-r} \min_{1 ≤ j ≤ n,i} \{ v_1(\binom{n}{j}) + qv_1(a_{r-1,i-j}) \} v_1(\binom{n}{i}) + qv_1(α) \}
\]
\[
≥ n + q^{d+1-r} \min_{1 ≤ j ≤ n,i} \{ -q^{-1} \left( \frac{i-j}{q^d} + \frac{n}{q^d-1} \right) \} \}
\]
\[
≥ n + q^{d+1-r} \left( -q^{-1} \left( \frac{i-1}{q^d} + \frac{n}{q^d-1} \right) \right)
\]
\[
= n - i + 1 - \frac{q^d n}{q^d - 1}.
\]
For $r = d + 1$, we have
\[
v_1(\beta_{d+1,d+1,i}) = v_1(b_{1,i}) \geq \min_{1 \leq j \leq n,i} \{ n - j + qv_1(a_{d,i-j}), n - i + qv_1(\alpha) + v_1\left(\frac{n}{i}\right) \}
\]
\[
\geq \min_{1 \leq j \leq n,i} \{ n - j - q^d \left( \frac{i-j}{q^d} + \frac{n}{q^d - 1} \right), 0 \}
\]
\[
\geq n - i - \frac{q^d n}{q^d - 1}.
\]
Thus we conclude that $v_1(\sum_{r=2}^{d+1} \beta_{d+1,r,i}) \geq n - i - q^d n/(q^d - 1)$. By considering the Newton polygon of $f_i$, we have $v_1(a_{1,i}) \geq -i/q^d - n/(q^d - 1)$ for any root $a_{1,i}$ of $f_i$. For $2 \leq r \leq m \leq d$, we have
\[
v_1(\beta_{m,r,i}) = q^{m-r} v_1(b_{r,i}) \geq q^{m-r} \left( -q^{r-1} \left( \frac{i-1}{q^d} + \frac{n}{q^d - 1} \right) \right) = -q^{m-1} \left( \frac{i-1}{q^d} + \frac{n}{q^d - 1} \right)
\]
and
\[
v_1(\gamma_m a_{1,i}^{q^{m-1}}) = q^{m-1} v_1(a_{1,i}) \geq -q^{m-1} \left( \frac{i}{q^d} + \frac{n}{q^d - 1} \right).
\]
Thus we have
\[
v_1(a_{m,i}) = v_1(\gamma_m a_{1,i}^{q^{m-1}} + \sum_{r=2}^m \beta_{m,r,i}) \geq -q^{m-1} \left( \frac{i}{q^d} + \frac{n}{q^d - 1} \right).
\]
\[
\square
\]

The next proposition is proved by similar arguments as Proposition 6.4.

**Proposition 6.5.** There exists an element $\Omega_v = \Omega_v(t) = (\sum_i a_{i,i}(t - \lambda_i))_l \in K^{\text{sep}}[t]_v^{\times} = \prod_l K^{\text{sep}}[t - \lambda_l]^{\times}$ which satisfies the equation
\[
(6.1) \quad \sigma(\Omega_v) = (t - \theta)\Omega_v.
\]

For any $l \in \mathbb{Z}/d$, $0 \leq m \leq d - 1$ and $i \geq 0$, we have
\[
v_1(a_{l+m,i}) = \frac{q^m}{q^{ld}(q^d - 1)}.
\]
By Propositions 6.4 and 6.5, the infinite sums $L_{\alpha,n}(\theta)$ and $\Omega_v(\theta)$ converge.

**Example 6.6.** We define the Carlitz motive to be the $\varphi$-module $C$ whose underlying $K(t)_v$-vector space is $K(t)_v$ and on which $\varphi$ acts by
\[
\varphi(f) = (t - \theta)\sigma(f)
\]
for each $f \in C$. The equation (6.1) means that the element $\Omega_v$ in Proposition 6.5 is a period of $C$. If we write $\Omega_v = (\Omega_{v,l})_l = (\sum_i a_{i,i}(t - \lambda_i))_l$, then $[K_d(a_{1,0}, a_{1,1}, \ldots) : K_d] = \infty$ by Proposition 6.5. Thus $\Omega_{v,l}$ is transcendental over $K(t)_v = K_d((t - \lambda_i))$. Therefore we have that $\text{tr.deg}_{K(t)_v} \Lambda_l = 1$ and $\Gamma_C = \mathbb{G}_m$. 

31
7 Algebraic independence of formal polylogarithms

In this section, we prove the algebraic independence of certain “formal” polylogarithms. The proof of this theorem follows [5] and [10] closely. Let \((F, E, L)\) be a \(\sigma\)-admissible triple and \(t, \theta \in E\) distinct elements. Let \(n, r\) be positive integers and \(\alpha_1, \ldots, \alpha_r \in E\) fixed elements. Assume that \((F^\infty)_{\text{tor}} \neq F^\infty\), and there exist elements \(\Omega = (\Omega_l)_l \in \mathbb{L}^\infty\) and \(L_{\alpha_j, n} = (L_{\alpha_j, n, l})_l \in L\) for each \(j = 1, \ldots, r\) such that \(\sigma(\Omega) = (t-\theta)\Omega, \Omega_l\) is transcendental over \(E\) and \(\sigma(L_{\alpha_j, n}) = \sigma(\alpha_j) + L_{\alpha_j, n}/(t-\theta)^n\). In the \(\nu\)-adic settings, such elements actually exist if \(\alpha_1, \ldots, \alpha_r \in K^\infty\) (cf. Section 6). We set

\[
\Phi := \begin{bmatrix}
(t - \theta)^n \\
\sigma(\alpha_1)(t - \theta)^n & 1 \\
\vdots & \ddots & \ddots \\
\sigma(\alpha_r)(t - \theta)^n & 1
\end{bmatrix}
\quad \text{and} \quad
\Psi := \begin{bmatrix}
\Omega^n \\
\Omega^n L_{\alpha_1, n} & 1 \\
\vdots & \ddots & \ddots \\
\Omega^n L_{\alpha_r, n} & 1
\end{bmatrix}.
\]

Then we have \(\sigma \Psi = \Phi \Psi\). Therefore, if \(M\) is the \(\varphi\)-module over \(E\) corresponding to \(\Phi\), then \(M\) is \(L\)-trivial. This type of \(t\)-motive is considered in [5] and [10]. Note that in \(\infty\)-adic case, \(\Omega\) and \(L_{\alpha, n}\) are constructed explicitly, and \(L_{\alpha, n}(\theta)\) is the \(n\)-th Carlitz polylogarithm of \(\alpha\). We define \(\Gamma, \Gamma_M, Z, \Lambda_t, \ldots\) as in the previous sections for \(\Phi\) and \(\Psi\). In particular, we have

\[
\Lambda_t = E(\Omega^n, L_{\alpha, n, t}, \ldots, L_{\alpha, n, t}).
\]

Furthermore, we assume that, \(\Gamma(F)\) is Zariski dense in \(\Gamma\) or \(\Lambda_t/F\) is regular extension for each \(l\). Thus the natural immersion \(\Gamma \to \Gamma_M\) is an isomorphism by Theorem 5.11.

For each \(F\)-algebra \(R\), we set

\[
G(R) := \left\{ \begin{bmatrix}
* & 0 & \cdots & 0 \\
* & 1 & \cdots & \cdots \\
\vdots & \cdots & \ddots & \cdots \\
* & & & 1
\end{bmatrix} \in \text{GL}_{r+1}(R) \right\}.
\]

Then \(G\) is an algebraic group over \(F\) and we have a natural inclusion \(\Gamma \subset G\). Let \(X_0, \ldots, X_r\) be the coordinates of \(G\) such that the first column of a general element of \(G\) “is”

\[
\begin{bmatrix}
X_0 \\
X_1 & 1 \\
\vdots & \ddots & \ddots \\
X_r & & & 1
\end{bmatrix}.
\]

We have the exact sequence \(1 \to G_{\alpha}^r \to G \to G_m \to 1\), here \(G_{\alpha}^r\) is the subgroup scheme of \(G\) with coordinates \((X_1, \ldots, X_r)\) and \(G_m\) is the quotient of \(G\) given by the projection \((X_i) \mapsto X_0\). Let \(C \in \Phi M^E_{\nu}\) be the one-dimensional \(\varphi\)-module such that \(\varphi(f) = (t - \theta)\sigma(f)\) for each \(f \in C = E\). Then we have the following exact sequence:

\[
0 \to C^{\otimes n} \to M \to 1^r \to 0.
\]

Thus \(C^{\otimes n}\) is an object of \(T_M\) and we have the canonical surjection \(\pi: \Gamma \cong \Gamma_M \to \Gamma_{C^{\otimes n}} \cong \mathbb{G}_m\). We set \(V := \ker \pi\). Then we have the commutative diagram

\[
\begin{array}{ccccccccc}
1 & \to & V & \to & \Gamma & \gamma \to & \mathbb{G}_m & \to & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \to & G_{\alpha}^r & \to & G & \to & \mathbb{G}_m & \to & 1,
\end{array}
\]

32
where the rows are exact.

**Proposition 7.1.** The subgroup $V$ of $G^r_a$ is defined by linear forms in $X_1,\ldots,X_r$ with $F$ coefficients.

**Proof.** Let $T \subset \Gamma_{\bar{F}}$ be a maximal torus and $\bar{\pi}: \Gamma_{\bar{F}} \to G_{m,\bar{F}}$ be the base extension of $\pi$ to $\bar{F}$. Then we have $\dim T = 1$ and $\bar{\pi}_T: T \to G_{m,\bar{F}}$ is an isomorphism. Thus $d\bar{\pi}$ is non-trivial and so is $d\pi$. Hence we have the following exact sequence:

$$0 \longrightarrow \text{Lie } V \longrightarrow \text{Lie } \Gamma \longrightarrow \text{Lie } G_{m,\bar{F}} \longrightarrow 0.$$ 

Since $\Gamma$ and $G_m$ are smooth over $F$, we have the equalities $\dim F \text{Lie } \Gamma = \dim \Gamma$ and $\dim F \text{Lie } G_m = 1$. Thus we have the equality $\dim F \text{Lie } V = \dim V$. Therefore $V$ is smooth over $F$. Thus it is enough to show that the space $V(\bar{F})$ is a linear space defined over $F$.

Let $T = \left[\begin{array}{cc} 1 & 0 \\ v & I_r \end{array}\right] \in V(\bar{F})$ and $\alpha \in \bar{F}^\times$ be any elements. Since $\Gamma(\bar{F}) \to G_{m,\bar{F}}(\bar{F})$ is surjective, there exists an element $\gamma \in \Gamma(\bar{F})$ such that $\pi(\gamma) = \alpha$. Then we have

$$V(\bar{F}) \ni \gamma^{-1} \mu = \left[\begin{array}{cc} 1 & 0 \\ \alpha v & I_r \end{array}\right].$$

Thus $V(\bar{F})$ is a linear subspace of $G^r_a(\bar{F})$. Since $V$ is defined over $F$, $V$ is defined by linear forms in $X_1,\ldots,X_r$ with $F$ coefficients. \hfill $\square$

Since $V$ is smooth and $H^1(F,V) = 1$, we have the exact sequence

$$1 \longrightarrow V(F) \longrightarrow \Gamma(F) \longrightarrow G_m(F) \longrightarrow 1.$$ 

By the assumption on $F$, there exists an element $b_0 \in F^\times \setminus (F^\times)_{\text{tor}}$. By the above sequence, there exists an element

$$\gamma = \left[\begin{array}{ccc} b_0 \\ b_1 & 1 \\ \vdots & \ddots \\ b_r & \end{array}\right] \in \Gamma(F).$$

We fix such $b_0$ and $\gamma$. For each $F$-algebra $R$ and $a \in R^\times$, we set

$$\gamma_a := \left[\begin{array}{ccc} a \\ \frac{b_1}{b_0} (a-1) & 1 \\ \vdots & \ddots \\ \frac{b_r}{b_0} (a-1) & \end{array}\right].$$

Then for each $a,b \in R^\times$ and $m \in \mathbb{Z}$, we have $\gamma_a \gamma_b = \gamma_{ab}$ and $\gamma^m = \gamma_{b_0^m}$. Hence we have $\overline{\{\gamma\}} = (R \to \{\gamma_a | a \in R^\times\})$, a line in $\Gamma$. We set $\Gamma' := \langle V,\gamma \rangle \subset \Gamma$ and $s := r - \dim V$. We claim that $\Gamma' = \Gamma$. Indeed, let

$$(7.1) \quad F_i = \sum_{j=1}^{r} c_{i,j} X_j \in F[X_1,\ldots,X_r] \quad (i = 1,\ldots,s)$$
be linear forms defining \( V \). For each \( i \), we set
\[
G_i := (b_0 - 1)F_i(X_1, \ldots, X_r) - F_i(b_1, \ldots, b_r)(X_0 - 1) \in F[X_0, \ldots, X_r].
\]
Then we can verify that \( G_1, \ldots, G_s \) define \( \Gamma' \) in \( \text{GL}_{r+1} \) and \( \Gamma' \) is an algebraic group. Since \( V \subset \Gamma' \subset \Gamma \) and \( \Gamma' \to \mathbb{G}_m \) is surjective, we have \( \Gamma' = \Gamma \). Thus we have the following proposition:

**Proposition 7.2.** The algebraic group \( \Gamma \) is defined by the linear polynomials \( G_1, \ldots, G_s \) in \( \text{GL}_{r+1}/F \).

Since \( Z_E \cong \Gamma_E \) and \( Z \) is defined over \( E \), \( Z \) is defined by linear polynomials over \( E \), and there exists an \( E \)-valued point
\[
\xi = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_r \\ 1 \end{bmatrix} \in Z(E).
\]
We fix such \( \xi \). Then we have \( Z = \xi \cdot \Gamma_E \). Set \( f'_i := G_i(f_0, \ldots, f_r)_{0}^{0} - 1 \in E \) and \( H_i := G_i(X_0, \ldots, X_r) - X_0 f'_i \in E[X_0, \ldots, X_r] \). Then \( H_1, \ldots, H_s \) are defining polynomials for \( Z \).

If we set \( g_i := \sum_{j=1}^{r} c_{i,j} b_j \), then we have
\[
H_i = (b_0 - 1) \sum_{j=1}^{r} c_{i,j} X_j + g_i - (g_i + f'_i)X_0.
\]
Since \( \Psi_l \in Z(\Sigma_l) \) for each \( l \), we have
\[
(b_0 - 1) \sum_{j=1}^{r} c_{i,j} L_{\alpha_j,n,l} + g_l \Omega_l^{-n} - (g_i + f'_i) = 0
\]
for each \( l \) and \( i \). Set \( B := (c_{i,j})_{i,j} \in \text{Mat}_{s \times r}(F) \). By the definition of \( c_{i,j} \) (7.1), the rank of \( B \) is \( s = r - \dim V \).

\[
P = \begin{bmatrix} P_i \\ P_s \end{bmatrix} := \begin{bmatrix} (b_0 - 1)c_{1,1} & \ldots & (b_0 - 1)c_{1,r} & g_1 & -(g_1 + f'_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (b_0 - 1)c_{s,1} & \ldots & (b_0 - 1)c_{s,r} & g_s & -(g_s + f'_s) \end{bmatrix} \in \text{Mat}_{s \times (r+2)}(E),
\]
the coefficients matrix of the above equations. Then the rank of \( P \) is also \( s \). We interested in
\[
N_l := \langle L_{\alpha_1,n,l}, \ldots, L_{\alpha_r,n,l}, \Omega_l^{-n}, 1 \rangle_E \subset \Lambda_l.
\]
This is the image of the \( E \)-linear map
\[
\beta_l: E^{r+2} \to \Lambda_l; \ (x_1, \ldots, x_{r+2}) \mapsto \sum_{j=1}^{r} x_j L_{\alpha_j,n,l} + x_{r+1} \Omega_l^{-n} + x_{r+2}.
\]
Since \( P_i \in \ker \beta_l \) for each \( i \), we have the inequality \( \dim_E \ker \beta_l \geq s \). Thus we have \( \dim_E N_l \leq r + 2 - s = \dim V + 2 = \dim \Gamma + 1 = \text{tr.deg}_E \Lambda_{l'} + 1 \) for each \( l \) and \( l' \). On the other hand, it is clear that \( \dim_E N_l \geq \text{tr.deg}_E \Lambda_l = \text{tr.deg}_E \Lambda_{l'} \). Thus we have the following theorem:

34
Theorem 7.3. For each \( l \) and \( l' \), we have \( \text{tr.deg}_E \Lambda_{l'} \leq \dim_E N_l \leq \text{tr.deg}_E \Lambda_{l'} + 1 \).

Corollary 7.4. If \( L_{\alpha_1,n,l}, \ldots, L_{\alpha_r,n,l}, 1 \) are linearly independent over \( E \) for some \( l \), then \( L_{\alpha_1,n,l'}, \ldots, L_{\alpha_r,n,l'} \) are algebraically independent over \( E \) for each \( l' \).

Proof. Note that since \( \Lambda_{l'} = E(\Omega_{l'}^n, L_{\alpha_1,n,l'}, \ldots, L_{\alpha_r,n,l'}) \), we have \( \text{tr.deg}_E \Lambda_{l'} \leq r + 1 \). By the assumption, we have \( r + 1 \leq \dim_E N_l \leq r + 2 \).

Assume that \( \dim_E N_l = \text{tr.deg}_E \Lambda_{l'} + 1 \). By Theorem 7.3, we have \( \dim E N_l = \text{tr.deg}_E \Lambda_{l'} + 1 \). Thus we have \( \text{tr.deg}_E \Lambda_{l'} = r + 1 \) and \( \Omega_{l'}^n, L_{\alpha_1,n,l'}, \ldots, L_{\alpha_r,n,l'} \) are algebraically independent over \( E \).

On the other hand, assume that \( \dim_E N_l = r + 1 \). By the assumption, we can write \( \Omega_{l'}^n \) as a linear combination of \( L_{\alpha_1,n,l}, \ldots, L_{\alpha_r,n,l}, 1 \) over \( E \). In particular, we have \( \Omega_{l'}^n \in E(L_{\alpha_1,n,l}, \ldots, L_{\alpha_r,n,l}) \). Letting \( \sigma \) act on this relation, we have

\[
(t - \theta)^n \Omega_{l'}^n \in \sigma(E) \left( \sigma(\alpha_1) + \frac{L_{\alpha_1,n,l+1}}{(t - \theta)^n}, \ldots, \sigma(\alpha_r) + \frac{L_{\alpha_r,n,l+1}}{(t - \theta)^n} \right).
\]

Thus for each \( l' \), we have \( \Omega_{l'}^n \in E(L_{\alpha_1,n,l'}, \ldots, L_{\alpha_r,n,l'}) \) and \( \Lambda_{l'} = E(L_{\alpha_1,n,l'}, \ldots, L_{\alpha_r,n,l'}) \). By Theorem 7.3, we have \( \text{tr.deg}_E \Lambda_{l'} \geq \dim_E N_l - 1 = r \). Thus \( \text{tr.deg}_E \Lambda_{l'} = r \) and \( L_{\alpha_1,n,l'}, \ldots, L_{\alpha_r,n,l'} \) are algebraically independent over \( E \).

References