Chemical and resistance exponents for 4D simple random walk

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Consider a simple random walk $S = (S_n)_{n \geq 0}$ on $\mathbb{Z}^d$ started at the origin. Regard $S[0, n]$ as a random graph whose vertex set and edge set are given by $\{S_k \,|\, 0 \leq k \leq n\}$ and $\big\{\{S_k, S_{k+1}\} \,|\, 0 \leq k \leq n-1\big\}$.

In [1], the following three quantities are studied:

- $D_n =$ the graph (chemical) distance between the origin and S_n on $S[0, n]$,
- $R_n =$ the effective resistance between the origin and S_n on $S[0, n]$,
- $L_n =$ the length (the number of steps) of the loop-erasure of $S[0, n]$.

In contrast to substantial progress in L_n not only for $d = 2$ but also for $d = 3$ ([3], [4]), much less is known about D_n and R_n for both $d = 2$ and $d = 3$.

What about the four-dimensional case? (For $d = 1$ and $d \geq 5$, the problem is much simpler.) It is shown in [2] that $\mathbb{E}(L_n)$ is asymptotic to $cn(\log n)^{-\frac{1}{3}}$. In this talk, I will show that there exist constants $c_1, c_2 > 0$ such that

$$\lim_{n\to\infty} \frac{\mathbb{E}(D_n)}{c_1 n (\log n)^{-\frac{1}{2}}} = 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{\mathbb{E}(R_n)}{c_2 n (\log n)^{-\frac{1}{2}}} = 1.$$

(The exact values of c_1 and c_2 are not computed. Even $c_1 \neq c_2$ is not proven!)

After establishing a law of large numbers for D_n and R_n, I will also present some fluctuation results for $D_n - \mathbb{E}(D_n)$ and $R_n - \mathbb{E}(R_n)$. These results will be useful for research on random interlacements and random walks on $S[0, \infty)$ in four dimensions.

References

[1] K. Burdzy and G. Lawler. Rigorous exponent inequalities for random walks. *J. Phys. A: Math. Gen.* **23**, L23. 1990.

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[4] D. Shiraishi. Growth exponent for loop-erased random walk in three dimensions. *Ann. Probab.* **46**, 687-774. 2018.