On higher dimensional analogues of Slepian's operator in his bandwidth arguments*

Dedicated to the memory of the late Professor Takeshi Kotake

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Abstract

Investigations are focused on partial differential operators which commute with certain integral operators analogous to those arising from the communication engineering. An explicit self-adjoint operator is derived for the case of disks in the plane.

1 Introduction

Suppose Q and P are *bounded* open subsets in $\mathbb{R}^n$. Define an operator $\mathcal{K}$: $\mathbf{L}^2(Q) \to \mathbf{L}^2(Q)$:

$$\mathcal{K}u(x) = \frac{1}{(2\pi)^n} \int_P \mathrm{e}^{i\,x\cdot y} \left(\int_Q \mathrm{e}^{-i\,x'\cdot y}\, u(x')\, dx' \right) dy. \tag{1}$$

$\mathcal{K}$ is an integral operator:

$$\mathcal{K}u(x) = \int_Q K_{Q,P}(x, x')\, u(x')\, dx' \tag{2}$$

where the kernel is given by

$$K_{Q,P}(x, x') = \frac{1}{(2\pi)^n} \int_P \mathrm{e}^{i(x-x')\cdot y}\, dy, \quad x,\, x' \in Q. \tag{3}$$

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Problem 1 *Determine the entire spectral property of the operator $\mathcal{K}$.*

This problem is related to Shannon's sampling theorem, or to the Uncertainty Principle. In fact, in the 1-dimensional case $n = 1$, consider a signal $u(t)$ with finite energy, i.e., square summable on the real line. By a practical reason, discard outside a certain interval $Q = [-T, T]$, or replace $u(t)$ by $u_T(t)$ which vanishes for $|t| > T$. Compute its Fourier image $\widehat{u_T}(\tau)$. In an essay of recovering the original signal from $\widehat{u_T}$, however, frequencies outside a certain range $P = [-\Omega, \Omega]$ being truncated, and thus get the mapping

$$u(t) \;\to\; u_T(t) \;\to\; \widehat{u_T}(\tau) \;\to\; \left\{ \begin{array}{ll} \widehat{u_T}(\tau), & -\Omega \le \tau \le \Omega \\ 0, & |\tau| > \Omega \end{array} \right\} \;\to\; \mathcal{K}u(t)$$

from $\mathbf{L}^2([-T, T])$ into itself. In this case,

$$K_{Q,P}(t, s) = \frac{1}{\pi} \frac{\sin \Omega\,(t-s)}{t-s}, \quad -T < t,\ s < T.$$

A complete solution to Problem 1 for $n = 1$ was given by Slepian ([5], [6]. See also Daubechies [1].). Slepian's idea is based on the identity:

$$\begin{aligned} \frac{\partial}{\partial t}\left\{(T^2 - t^2)\frac{\partial}{\partial t}\frac{\sin \Omega\,(t-s)}{t-s}\right\} - \frac{\partial}{\partial s}\left\{(T^2 - s^2)\frac{\partial}{\partial s}\frac{\sin \Omega\,(t-s)}{t-s}\right\} \\ = \Omega^2\,(t^2 - s^2)\,\frac{\sin \Omega\,(t-s)}{t-s}. \end{aligned} \tag{4}$$

He was thus able to reduce the spectral problem of $\mathcal{K}$ to that of the Mathieu-like ordinary differential operator

$$\frac{d}{dt}\Big((T^2 - t^2)\frac{d}{dt}\,\cdot\,\Big) - \Omega^2\,t^2 \tag{5}$$

which commutes with the integral operator $\mathcal{K}$ on the interval $[-T, T]$. The eigenfunctions of $\mathcal{K}$ are fully described with the prolate spheroidal wave functions (Consult, e.g., [3]). He applied this result to provide a *rigorous* interpretation of the Nyquist condition of the sampling theory (the WT theorem).

In the present note, we derive partial differential operators as analogues to the operator (5) which ensure the commutation relations corresponding to (4). In particular, we specify one in the case when Q and P are disks in the plane. We also show that the thus obtained operator is self-adjoint in the space $\mathbf{L}^2(Q)$.

2 Basics

Now we are back to the case $n \geq 2$. Here are some of the basic properties of the integral operator $\mathcal{K}$ in the context of Functional Analysis.

Lemma 1 *$\mathcal{K}$ is a bounded self-adjoint strictly positive operator in $\mathbf{L}^2(D)$. The maximum eigenvalue of $\mathcal{K}$ is smaller than 1.*

In fact, for $u(x) \in \mathbf{L}^2(D)$, we have

$$\int_Q \mathcal{K}u(x)\,\overline{u(x)}\,dx = \int_P \left|\mathfrak{F}\Big(c_Q u\Big)(y)\right|^2 dy = \int_Q u(x)\,\overline{\mathcal{K}u(x)}\,dx. \tag{6}$$

Here $\mathfrak{F}$ denotes the Fourier transform

$$\mathfrak{F}\varphi(y) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \mathrm{e}^{-ix\cdot y}\,\varphi(x)\,dx, \quad \varphi(x) \in \mathbf{L}^2(\mathbb{R}^n),$$

and $c_Q(x)$ is the charactersitic function of the set Q. The quadratic form (6) vanishes only when $c_Q(x)u(x) = 0$ in $\mathbf{L}^2(\mathbb{R}^n)$, i.e., when $u(x) = 0$ in $\mathbf{L}^2(Q)$. On the other hand, let μ_0 is the maximum eigenvalue of $\mathcal{K}$. Then

$$\mu_0 = \sup \frac{\int_Q \mathcal{K}u(x)\,\overline{u(x)}\,dx}{\int_Q |u(x)|^2\,dx} = \sup \frac{\int_P \left|\mathfrak{F}\Big(c_Q u\Big)(y)\right|^2 dy}{\int_{\mathbb{R}^n} \left|\mathfrak{F}\Big(c_Q u\Big)(y)\right|^2 dy} < 1.$$

For, otherwise, we would have a contradiction that $\mathfrak{F}\Big(c_Q u\Big)(y) = 0$ outside P for some $c_Q(x)u(x) \neq 0$.

Lemma 2 *The operator $\mathcal{K} : \mathbf{L}^2(Q) \to \mathbf{L}^2(Q)$ is compact.*

In fact, the kernel $K_{Q,P}(x, x')$ is of class C^∞ in x, x' (actually in $x - x'$) as is obvious from (3). We have[1]

$$\partial_x^\alpha\, K_{Q,P}(x, x') = (-1)^{|\alpha|}\partial_{x'}^\alpha K_{Q,P}(x, x') = \frac{1}{(2\pi)^n} \int_P i^{|\alpha|}\, y^\alpha\, \mathrm{e}^{i(x-x')\cdot y}\, dy,$$

whence

$$|\partial_x^\alpha\, K_{Q,P}(x, x')| = |\partial_{x'}^\alpha\, K_{Q,P}(x, x')| \leq \frac{|P|}{(2\pi)^n}\mathrm{diameter}(P)^{|\alpha|}.$$

[1] $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$ is a multi-index. $|\alpha| = \alpha_1 + \cdots + \alpha_n$ is the length of α, and

$$\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \quad \left(\partial_j = \frac{\partial}{\partial x_j}\right), \quad y^\alpha = y_1^{\alpha_1} \cdots y_n^{\alpha_n}$$

for $\partial = (\partial_1, \cdots, \partial_n)$ and $y = (y_1, \cdots, y_n)$.

(3) shows more. We have an entire analytic function

$$F_P(\zeta) = \frac{1}{(2\pi)^n} \int_P e^{i\zeta\cdot y}\, dy, \quad \zeta \in \mathbb{C}^n, \tag{7}$$

such that $K_{Q,P}(x, x') = F_P(x - x')$, x, $x' \in Q$. Therefore, for any $u \in \mathbf{L}^2(Q)$, $\mathcal{K}u(x)$ is bounded in Q and of class C^ω in Q.

Corollary 1 *The kernel is skew-symmetric:* $K_{Q,P}(x, x') = \overline{K_{Q,P}(x', x)}$. $K_{Q,P}(x, x')$ *takes real values if* P *is symmetric with respect to the origin :* $-P = P$.

Example 1 Let $n = 2$ and P the disk $\{y_1^2 + y_2^2 < \sigma^2\}$ $(\sigma > 0)$. Then

$$F_P(\zeta) = \frac{1}{|\zeta|^2} \frac{1}{2\pi} \int_0^{\sigma|\zeta|} J_0(r)\, r dr = \frac{\sigma}{2\pi} \frac{J_1(\sigma\,|\zeta|)}{|\zeta|}$$

for *real* $\zeta \in \mathbb{R}^2$, $\zeta \neq 0$. Hence, in this case,

$$K_{Q,P}(x, x') = \begin{cases} \dfrac{\sigma}{2\pi} \dfrac{J_1(\sigma|x - x'|)}{|x - x'|}, & x \neq x' \\ \dfrac{1}{2\pi}\sigma^2, & x = x' \end{cases} \tag{8}$$

for $x = (x_1, x_2)$, $x' = (x_1', x_2') \in Q$ and $|x - x'| = \sqrt{(x_1 - x_1')^2 + (x_2 - x_2')^2}$.

Example 2 Let $n = 3$ and P the ball $\{y_1^2 + y_2^2 + y_3^2 < \sigma^2\}$ $(\sigma > 0)$. Then

$$F_P(\zeta) = \frac{1}{(2\pi)^2} \int_0^\sigma r^2\, dr \left\{ \int_0^\pi J_0\left(r\sqrt{\zeta_1^2 + \zeta_2^2}\ \sin\psi \right) e^{ir\zeta_3 \cos\psi} \sin\psi\, d\psi \right\}$$

for $\zeta \in \mathbb{R}^3 \setminus \{0\}$, as seen by the introduction of the spherical coordinates. The interior integral turns out to be

$$\sqrt{\frac{2\pi}{r|\zeta|}}\, J_{1/2}(r\,|\zeta|) = 2\,\frac{\sin r|\zeta|}{r\,|\zeta|},$$

whence

$$F_P(\zeta) = \sigma^3\, \frac{J_{3/2}(\sigma\,|\zeta|)}{\sqrt{2\pi\sigma|\zeta|}^3}$$

(See [2], formula (4), p.46). Actually, this argument is extended for $n = 4, 5, \cdots$. Thus, for any $n = 1, 2, \cdots$, we have

$$F_P(\zeta) = \sigma^n\, \frac{J_{n/2}(\sigma\,|\zeta|)}{\sqrt{2\pi\sigma|\zeta|}^n}, \quad \zeta \in \mathbb{R}^n \setminus \{0\}, \tag{9}$$

when P is the ball of radius σ in $\mathbb{R}^n$ centered at the origin.

Lemmas 1 and 2 imply that the spectrum $\sigma(\mathcal{K})$ of $\mathcal{K}$ consists of discrete positive eigenvalues μ_k, $k = 0, 1, 2, \cdots$ (of finite multiplicity) with $1 > \mu_0 > \mu_1 > \cdots \to 0$ (See, e.g., [8], Chapter X).

Remark 2.1 The function $F_P(\zeta)$ of (7) is of course an analogy to Whittaker's sinc function. Note

$$|\zeta_k \, F_P(\zeta)| \le |\partial P| \, \exp(\sup_{y\in\partial P} \Im\zeta \cdot y), \quad k = 1, \cdots, n,$$

by Stokes' theorem.

Note also that the equation $u - \mathcal{K}u = f$ for a given $f \in \mathbf{L}^2(Q)$ is uniquely solved by $u = \sum_{m=0}^{\infty} \mathcal{K}^m f \in \mathbf{L}^2(Q)$. Here each $\mathcal{K}^m$ turns out an integral operator with the kernel

$$K^{(m)}_{Q,P}(x, x') = \int_Q dx^{(1)} \cdots \int_Q dx^{(m-1)} \, \big\{K_{Q,P}(x, x^{(1)}) \cdots K_{Q,P}(x^{(m-1)}, x')\big\}$$

for $m = 2, 3, \cdots$ ($K^{(1)}_{Q,P}(x, x') = K_{Q,P}(x, x')$). We can just rewrite

$$K^{(m)}_{Q,P}(x, x') = \frac{1}{(2\pi)^n} \int_P dy \int_P dy' \, \mathrm{e}^{i(x\cdot y - x'\cdot y')} \, k_m(y, y') \tag{10}$$

with

$$k_2(y, y') = F_Q(y' - y) = \frac{1}{(2\pi)^n} \int_Q \mathrm{e}^{ix''\cdot(y'-y)} \, dx'' \tag{11}$$

and

$$\begin{aligned} k_{m+1}(y, y') =& \frac{1}{(2\pi)^n} \int_Q dx'' \int_P dy'' \, \mathrm{e}^{ix''\cdot(y''-y)} \, k_m(y'', y') \\ =& \int_P F_Q(y'' - y) \, k_m(y'', y') \, dy''. \end{aligned}$$

Hence,

$$k_m(y, y') = \int_P dy^{(2)} \cdots \int_P dy^{(m-1)} \, F_Q(y' - y^{(2)}) \cdots F_Q(y^{(m-1)} - y) \tag{12}$$

for $m = 3, 4, \cdots$.

Remark 2.2 Note $K_{Q,F}(x, x) = F_P(0) = (2\pi)^{-n} \, |P|$. Thus,

$$\kappa_1 = \int_Q K_{Q,P}(x, x) \, dx = \frac{|P| \, |Q|}{(2\pi)^n}.$$

Also we have

$$K^{(m)}_{Q,P}(x,x) = \frac{1}{(2\pi)^n} \int_P dy \int_P dy' \, \mathrm{e}^{ix\cdot(y-y')} \, k_m(y,y'),$$

whence

$$\begin{aligned}\kappa_m &= \int_Q K^{(m)}_{Q,P}(x,x)\, dx = \int_P dy \int_P dy' \, F_Q(y-y')\, k_m(y,y') \\ &= \int_P dy^{(1)} \cdots \int_P dy^{(m)} \times \\ &\qquad \times F_Q(y^{(1)}-y^{(2)}) \cdots F_Q(y^{(m-1)}-y^{(m)})\, F_Q(y^{(m)}-y^{(1)})\end{aligned}$$

for $m = 2, 3, \cdots$.

To discuss the spectral property of the operator $\mathcal{K}$ in somewhat general context, a computation of the Fredholm determinant might be useful. Let $\Delta_m(x^{(1)}, \cdots, x^{(m)})$ be the determinant of the $m \times m$-matrix with the (j,k)-entry $K_{Q,P}(x^{(j)}, x^{(k)})$. Thus, for instance, $\Delta(x^{(1)}) = K_{Q,P}(x^{(1)}, x^{(1)}), \cdots$,

$$\Delta_3(x^{(1)}, x^{(2)}, x^{(3)}) = \begin{vmatrix} K_{Q,P}(x^{(1)}, x^{(1)}) & K_{Q,P}(x^{(1)}, x^{(2)}) & K_{Q,P}(x^{(1)}, x^{(3)}) \\ K_{Q,P}(x^{(2)}, x^{(1)}) & K_{Q,P}(x^{(2)}, x^{(2)}) & K_{Q,P}(x^{(2)}, x^{(3)}) \\ K_{Q,P}(x^{(3)}, x^{(1)}) & K_{Q,P}(x^{(3)}, x^{(2)}) & K_{Q,P}(x^{(3)}, x^{(3)}) \end{vmatrix}.$$

Let

$$A_m = (-1)^m \int_Q dx^{(1)} \cdots \int_Q dx^{(m)} \, \Delta_m(x^{(1)}, \cdots, x^{(m)})$$

for $m = 1, 2, \cdots$. Then

$$\begin{gathered} A_1 = -\kappa_1,\ A_2 = \kappa_1^2 - \kappa_2,\ A_3 = -\kappa_1^3 + 3\kappa_1\kappa_2 - 2\kappa_3, \\ A_4 = \kappa_1^4 - 6\kappa_1^2\kappa_2 + 8\kappa_1\kappa_3 + 3\kappa^2 - 6\kappa_4,\ \cdots . \end{gathered}$$

Now put

$$\Delta(z) = 1 + \sum_{m=1}^{\infty} \frac{A_m}{m!} z^m, \quad z \in \mathbb{C}. \tag{13}$$

$\Delta(z)$ is an entire analytic function of $z \in \mathbb{C}$. In fact, it is well-known that the series on the right-hand side uniformly converges in z on any bounded set in $\mathbb{C}$ (Consult, e.g., [7], Chapter XI).

Let $\Phi_m(x, x''; x^{(1)}, \cdots, x^{(m-1)})$ be the determinant of the $m \times m$-matrix with the (j,k)-entry $K_{Q,P}(x^{(j-1)}, x^{(k-1)})$ for $j,\, k \geq 2$ while $K_{Q,P}(x, x'')$,

$K_{Q,P}(x, x^{(k-1)})$ and $K_{Q,P}(x^{(k-1)}, x'')$, repectively, the $(1,1)$-, $(1,k)$- and $(k,1)$-entries, $(k \geq 2)$. Thus, e.g.,

$$\Phi_3(x, x''; x^{(1)}, x^{(2)}) = \begin{vmatrix} K_{Q,P}(x, x'') & K_{Q,P}(x, x^{(1)}) & K_{Q,P}(x, x^{(2)}) \\ K_{Q,P}(x^{(1)}, x'') & K_{Q,P}(x^{(1)}, x^{(1)}) & K_{Q,P}(x^{(1)}, x^{(2)}) \\ K_{Q,P}(x^{(2)}, x'') & K_{Q,P}(x^{(2)}, x^{(1)}) & K_{Q,P}(x^{(2)}, x^{(2)}) \end{vmatrix}.$$

Let then

$$\begin{aligned} &\Psi_m(x, x') \\ &= -\, m \int_Q dx'' \int_Q dx^{(1)} \cdots \int_Q dx^{(m-1)}\, K_{Q,P}(x'', x')\, \Phi_m(x, x''; x^{(1)}, \cdots, x^{(m-1)}) \end{aligned}$$

and finally put

$$\Delta(x, x'; z) = z\, \Delta(z)\, K_{Q,P}(x, x') + \sum_{m=1}^{\infty} (-1)^m \frac{\Psi_m(x, x')}{m!}\, z^{m+1}. \qquad (14)$$

$\Delta(x, x'; z)$ is again an entire analytic function of $z \in \mathbb{C}$ as is the case of $\Delta(z)$. We have

$$\Delta(x, x'; z) = z\, \Delta(z)\, K_{Q,P}(x, x') + z \int_Q \Delta(x, x''; z)\, K_{Q,P}(x'', x')\, dx''$$

for x, $x' \in Q$ and $z \in \mathbb{C}$.

Lemma 3 *Let $f(x) \in \mathbf{L}^2(Q)$. If $\Delta(z) \neq 0$, then*

$$u(x) = f(x) + \int_Q \frac{\Delta(x, x'; z)}{\Delta(z)}\, f(x')\, dx' \in \mathbf{L}^2(Q) \qquad (15)$$

and $u(x)$ solves the equation $u - z\,\mathcal{K}u = f$ in $\mathbf{L}^2(Q)$.

Those $z \in \mathbb{C}$ with $\Delta(z) = 0$ are called characteristic values of the kernel $K_{Q,P}(x, x')$ of $\mathcal{K}$. Characteristic values are reciprocals of eigenvalues mentioned in Lemma 1. If μ_m is the m-th eigenvalue of the operator $\mathcal{K}$, then $z_m = 1/\mu_m$ is the m-th characteristic value of the kernel.

Note (15) can be expressed in the operator form as

$$u = (I - z\,\mathcal{K})^{-1} f, \quad \Delta(z) \neq 0.$$

Recall $(\lambda I - \mathcal{K})^{-1} = \frac{1}{\lambda}\,(I - \frac{1}{\lambda}\,\mathcal{K})^{-1}$ for $\lambda \notin \sigma(\mathcal{K})$. For each eigenvalue μ_m of $\mathcal{K}$, consider the operator

$$\mathcal{E}_m = \frac{1}{2\pi i} \int_{\gamma_m} (\lambda I - \mathcal{K})^{-1}\, d\lambda.$$

Here γ_m is a small circle in $\mathbb{C}$ centered at μ_m, which contains no other points in the spectrum $\sigma(\mathcal{K})$. $\mathcal{E}_m$ is in fact the projection operator onto the eigenspace corresponding to the eigenvalue μ_m. Thus, let

$$E(x, x'; z_m) = -\frac{1}{2\pi i} \int_{\gamma'_m} \frac{\Delta(x, x'; z)}{\Delta(z)} \frac{dz}{z},$$

where γ'_m is a small circle centered at $z_m = 1/\mu_m$, which contains no other zeroes than z_m of $\Delta(z)$. Then $\mathcal{E}_m$ is represented as an integral operator:

$$\mathcal{E}_m \, : \, f(x) \ \mapsto \int_Q E(x, x'; z_m)\, f(x')\, dx'.$$

Thus, Problem 1 is reduced to *explicit* investigations of $\Delta(x, x'; z)/\Delta(z)$.

3 Analogy

Now we try to follow Slepian's approach ([5]) and establish an analogy to the identity (4). Thus, we check when the integral operator $\mathcal{K}$ commute with a self-adjoint operator $\mathcal{A}$ realized by a second order partial differential operator $a(x, D)$. It turns out these premises impose quite a restrictive relation among the domains Q, P and the operator $a(x, D)$ (See Proposition 1 below).

To begin with, consider a second order partial differential operator:

$$a(x, D) \cdot = \sum_{j,\,k=1}^{n} D_j \Big(a_{jk}(x)\, D_k \cdot \Big) + c(x) \cdot , \quad D_j = \frac{1}{i}\, \partial_j, \tag{16}$$

and its symbol

$$a(x, y) = \sum_{j,\,k=1}^{n} a_{jk}(x)\, y_j y_k + c(x). \tag{17}$$

Here $a_{jk}(x)$, j, $k = 1, \cdots, n$, and $c(x)$ are real-valued bounded and continuous functions on the closure $\overline{Q}$, sufficiently smooth in Q. Assume further $a_{jk}(x) = a_{kj}(x)$. The homogeneous term of the highest, i.e., second, degree is denoted

$$a_2(x, y) = \sum_{j,\,k=1}^{n} a_{jk}(x)\, y_j y_k.$$

Lemma 4 *The operator $a(x, D)$ is symmetric, that is,*

$$\int_Q a(x, D) v(x)\, \overline{w(x)}\, dx = \int_Q v(x)\, \overline{a(x, D) w(x)}\, dx$$

holds for $v(x),\, w(x) \in C_0^2(Q)$. *In particular,*

$$\int_Q a(x,D)v(x)\,v(x)\,dx = \int_Q a_2(x, \partial\, v(x))\,dx + \int_Q c(x)\,v(x)^2\,dx$$

for a real-valued $v(x) \in C_0^2(D)$.

Proof. Suppose $v(x)$ and $w(x)$ are smooth in Q and vanish on the boundary ∂Q. Then

$$\int_Q a(x,D)v(x)\,\overline{w(x)}\,dx - \int_Q v(x)\,\overline{a(x,D)w(x)}\,dx$$
$$= \sum_{j=1}^{n}\sum_{k=1}^{n} \int_Q D_j\Big(\,a_{jk}(x)\,D_k v(x)\cdot \overline{w(x)} + v(x)\cdot a_{jk}(x)\overline{D_k w(x)}\Big)\,dx.$$

the right-hand side vanishes because of Stokes' theorem and assumtions on $v(x)$ and $w(x)$. Q.E.D.

We wish to extend the operator $a(x,D)$ as a self-adjoint operator $\mathcal{A}$ in $\mathbf{L}^2(Q)$ by supplying an appropriate boundary condition on ∂Q.

However, self-adjoint extensions are not necessarily unique. We also wish to have one which commutes with the operator $\mathcal{K}$, that is, for which $\mathcal{A}\,\mathcal{K} - \mathcal{K}\,\mathcal{A} = 0$ holds on the domain of $\mathcal{A}$.

Actually, to achieve an analogy to (4), we here will specify that $a_{jk}(x)$ and $c(x)$ are polynomials in x of degree 2. However, the requirement that the involved polynomials be of degree 2may be too restrictive in the general context as Proposition 4 below suggests.

Thus, we assume

$$a_{jk}(x) = \sum_{p=1}^{n}\sum_{q=1}^{n} a_{jkpq} x_p x_q + b_{jk}, \quad j,\, k = 1, \cdots, n, \tag{18}$$

and

$$c(x) = \sum_{p=1}^{n}\sum_{q=1}^{n} c_{pq}\, x_p x_q. \tag{19}$$

Here $a_{jkpq} = a_{jkqp} = a_{kjpq}$, $b_{jk} = b_{kj}$ and $c_{pq} = c_{qp}$. In particular, if we put

$$A(x,y) = \sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{p=1}^{n}\sum_{q=1}^{n} a_{jkpq}\, x_p x_q y_j y_k \tag{20}$$

and

$$b(y) = \sum_{j=1}^{n}\sum_{k=1}^{n} b_{jk} y_j y_k, \tag{21}$$

then the symbol $a(x,y)$ turns out

$$a(x,y) = A(x,y) + b(y) + c(x). \tag{22}$$

Proposition 1 *Assume that the symbol $a(x,y)$ is given by (22) with (20) (21) and (19). Suppose the boundaries ∂Q of Q and ∂P of P are sufficiently regular, e.g., of class C^1. Let $\nu(x) = (\nu_1(x), \cdots, \nu_n(x))$ be the unit outer normal at $x \in \partial Q$ and $dS(x)$ the surface area. Let $\mu(x) = (\mu_1(x), \cdots, \mu_n(x))$ be the unit outer normal at $y \in \partial P$ and $d\Sigma(y)$ the surface area. If*

$$\sum_{j=1}^{n} \Big\{ \sum_{p=1}^{n} \sum_{q=1}^{n} a_{jkpq} x_p x_q + b_{jk} \Big\} \nu_j(x) = 0, \quad k = 1, \cdots, n, \tag{23}$$

on ∂Q and

$$\sum_{p=1}^{n} \Big\{ \sum_{j=1}^{n} \sum_{k=1}^{n} a_{jkpq} y_j y_k + c_{pq} \Big\} \mu_p(y) = 0, \quad q = 1, \cdots, n \tag{24}$$

on ∂P, then the operators $a(x,D)$ and $\mathcal{K}$ commute, that is,

$$a(x,D)\Big(\mathcal{K}u\Big)(x) = \mathcal{K}\Big(a(\cdot,D)u\Big)(x), \quad u \in \mathbf{L}^2(Q) \cap C^2(\mathbb{R}^n)$$

holds.

Therefore, in order to ensure an analogy to (4), we need a rather stringent looking requirements (23) (24) on the partial differential operator $a(x,D)$ of (16) and the domains Q and P.

Remark 3.1 The condition (23) implies

$$\det\Big(\sum_{p=1}^{n} \sum_{q=1}^{n} a_{jkpq} x_p x_q + b_{jk} \Big) = 0 \tag{25}$$

and $a_2(x, \nu(x)) = 0$ on the boundary ∂Q. On the other hand, (24) implies $a(\mu(y), y) = b(y)$ and

$$\det\Big(\sum_{j=1}^{n} \sum_{k=1}^{n} a_{jkpq} y_j y_k + c_{pq} \Big) = 0 \tag{26}$$

on ∂P.

To verify Proposition 1, we discuss in a slightly more general hypothesis than that made in the proposition. Note

$$a(x,D)\,K_{Q,P}(x,x') = \frac{1}{(2\pi)^n}\int_P a(x,y)\,\mathrm{e}^{i(x-x')\cdot y}\,dy + \frac{1}{(2\pi)^n}\int_P \Big(\frac{1}{i}\sum_{j=1}^{n}\sum_{k=1}^{n}\partial_j a_{jk}(x) y_k\Big)\mathrm{e}^{i(x-x')\cdot y}\,dy$$

by a straight forward computation.

On the other hand, for $u(x)\in \mathbf{L}^2(Q)\cap C^2(\mathbb{R}^n)$, consider[2]

$$\begin{aligned}&\int_Q K_{Q,P}(x,x')\,a(x',D')u(x')\,dx'\\ &= \sum_{j=1}^{n}\int_Q D'_j\Big(\sum_{k=1}^{n} K_{Q,P}(x,x')\,a_{jk}(x')D'_k u(x')\Big)\,dx'\\ &\quad - \sum_{k=1}^{n}\int_Q D'_k\Big(\sum_{j=1}^{n} D'_j K_{Q,P}(x,x')\cdot a_{jk}(x')u(x')\Big)\,dx'\\ &\quad + \int_Q \Big(a(x',D')\,K_{Q,P}(x,x')\Big)\,u(x')\,dx.\end{aligned} \tag{27}$$

The first two sums on the right-hand side will be reduced to integrals on the boundary ∂Q by Stokes' theorem provided ∂Q enjoys an adequate regularity.

Lemma 5 *Suppose ∂Q is regular enough. Then the first two sums on the right-hand side of (27) turn out*

$$-\int_{\partial Q} K_{Q,P}(x,x')\sum_{k=1}^{n}\Big(\sum_{j=1}^{n} a_{jk}(x')\,\nu_j(x')\Big)\partial'_k u(x')\,dS(x') + \int_{\partial Q}\sum_{k=1}^{n}\Big(\sum_{j=1}^{n} a_{jk}(x')\,\nu_j(x')\Big)\partial'_k K_{Q,P}(x,x')\cdot u(x')\,dS(x').$$

In particular, if (23) holds, then all these integrals vanish.

The third term on the right-hand side of (27) has the kernel

$$a(x',D')K_{D,\Delta}(x,x') = \frac{1}{(2\pi)^n}\int_P a(x',-y)\,\mathrm{e}^{i(x-x')\cdot y}\,dy + \frac{1}{(2\pi)^n}\int_P \Big(-\frac{1}{i}\sum_{j=1}^{n}\sum_{k=1}^{n}\partial'_j a_{jk}(x') y_k\Big)\,\mathrm{e}^{i(x-x')\cdot y}\,dy.$$

[2]Here $\partial' = (\partial'_1,\cdots,\partial'_n)$, where $\partial'_j = \dfrac{\partial}{\partial x'_j}$. Thus, $D'_j = \frac{1}{i}\,\partial'_j$.

Therefore,

$$a(x,D)\,K_{Q,P}(x,x') - a(x',D')\,K_{Q,P}(x,x') \\ = \frac{1}{(2\pi)^n}\int_P r(x,x',y)\,\mathrm{e}^{i(x-x')\cdot y}\,dy \tag{28}$$

where

$$r(x,x',y) == a_2(x,y) - a_2(x',y) + c(x) - c(x') \\ + \frac{1}{i}\sum_{k=1}^{n}\sum_{j=1}^{n}\Big\{\partial_j a_{jk}(x) + \partial'_j a_{jk}(x')\Big\} y_k.$$

Note

$$\frac{1}{i}\sum_{k=1}^{n}\sum_{j=1}^{n}\partial_j a_{jk}(x) y_k = -\frac{i}{2}\sum_{k=1}^{n} a_{2(k)}^{(k)}(x,y)$$

where $a_{2(j)}^{(k)}(x,y) = \frac{\partial^2}{\partial x_j \partial y_k} a_2(x,y)$. Hence, then

$$r(x,x',y) \\ = a_2(x,y) - a_2(x',y) - \frac{i}{2}\sum_{k=1}^{n}\Big(a_{2(k)}^{(k)}(x,y) + a_{2(k)}^{(k)}(x',y)\Big) + c(x) - c(x'). \tag{29}$$

Now use (20) (22) and (19). We have

$$\begin{aligned} a_2(x,y) =& a_2(x',y) + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{p=1}^{n}\sum_{q=1}^{n} a_{jkpq}(x_p + x'_p)(x_q - x'_q) y_j y_k \\ =& a_2(x',y) + \frac{1}{2}\sum_{q=1}^{n} a_{2(q)}(x+x',y)\,(x_q - x'_q), \\ c(x) =& c(x') + \frac{1}{2}\sum_{p=1}^{n}\sum_{q=1}^{n} c_{pq}(x_p + x'_p)(x_q - x'_q) \\ =& c(x') + \frac{1}{2}\sum_{q=1}^{n} c_{(q)}(x+x')\,(x_q - x'_q). \end{aligned}$$

It follows

$$r(x,x',y)\,\mathrm{e}^{i(x-x')\cdot y} = \frac{1}{2i}\sum_{p=1}^{n}\hat{\partial}_p\left\{\Big(a_{2(p)}(x+x',y) + c_{(p)}(x+x')\Big)\,\mathrm{e}^{i(x-x')\cdot y}\right\}$$

where $\hat{\partial}_p = \partial/\partial y_p$. Hence, when the boundary ∂P is regular enough,

$$\begin{aligned}&\int_P r(x,x',y)\,\mathrm{e}^{i(x-x')\cdot y}\,dy\\ =&\frac{1}{2i}\int_{\partial P}\sum_{p=1}^{n}\mu_p(y)\left(a_{2(p)}(x+x',y)+c_{(p)}(x+x')\right)\mathrm{e}^{i(x-x')\cdot y}\,d\Sigma(y)\end{aligned} \tag{30}$$

where $\mu(y) = (\mu_1(y), \cdots, \mu_n(y))$ is the unit outer normal and $d\Sigma(y)$ the surface area at $y \in \partial P$. The right-hand side of (30) vanishes when (24) is satisfied.

Hence, we have the following

Lemma 6 *Suppose the boundary ∂P of P is regular enough, e.g., of class C^1. Let $\mu(y) = (\mu_1(y), \cdots, \mu_n(y))$ be the unit outer normal and $d\Sigma(y)$ the surface area at $y \in \partial P$. Assume (22) with (20) and (19). If (24) holds, then*

$$a(x,D)\,K_{Q,P}(x,x') - a(x',D')\,K_{Q,P}(x,x') = 0.$$

Proposition 1 is now proved.

4 An analysis of the 2 dimensional case

Let us analyze the conditions (23) and (24) for the planar case. Observe (25) and (26) take the following forms:

$$\begin{aligned}&\Big\{(a_{1111}a_{2211}-a_{2111}^2)x_1^4+2(a_{1111}a_{2221}+a_{1121}a_{2211}-2a_{2111}a_{2121})x_1^3x_2\\ &+(a_{1111}a_{2222}+a_{1122}a_{2211}-2a_{2111}a_{2122}+4a_{1121}a_{2221}-4a_{2121}^2)x_1^2x_2^2\\ &+2(a_{1121}a_{2222}+a_{1122}a_{2221}-2a_{2121}a_{2122})x_1x_2^3+(a_{1122}a_{2222}-a_{2122}^2)x_2^4\Big\}\\ &+\Big\{(a_{1111}b_{22}-2a_{2111}b_{21}+a_{2211}b_{11})x_1^2\\ &+(a_{1122}b_{22}-2a_{2122}b_{21}+a_{2222}b_{11})x_2^2\\ &+2(a_{1121}b_{22}-2a_{2121}b_{21}+a_{2221}b_{11})x_1x_2\Big\}+(b_{11}b_{22}-b_{21}^2)=0\end{aligned} \tag{31}$$

and

$$\begin{aligned}&\Big\{(a_{1111}a_{1122}-a_{1121}^2)y_1^4+2(a_{1111}a_{2122}+a_{2111}a_{1122}-2a_{1121}a_{2121})y_1^3y_2\\&+(a_{1111}a_{2222}+a_{1122}a_{2211}-2a_{1121}a_{2221}+4a_{2111}a_{2122}-4a_{2121}^2)y_1^2y_2^2\\&+2(a_{2111}a_{2222}+a_{2211}a_{2122}-2a_{2121}a_{2221})y_1y_2^3+(a_{2211}a_{2222}-a_{2221}^2)y_2^4\Big\}\\&+\Big\{(a_{1111}c_{22}-2a_{1121}c_{21}+a_{1122}c_{11})y_1^2\\&+(a_{2211}c_{22}-2a_{2221}c_{21}+a_{2222}c_{11})y_2^2\\&+2(a_{2111}c_{22}-2a_{2121}c_{21}+a_{2122}c_{11})y_1y_2\Big\}+(c_{11}c_{22}-c_{21}^2)=0\end{aligned} \tag{32}$$

respectively on ∂Q and on ∂P. Then (23), for instance, implies

$$\begin{aligned}&\Big(a_{2111}x_1^2+2a_{2121}x_1x_2+a_{2122}x_2^2+b_{21}\Big)dx_1\\-&\Big(a_{1111}x_1^2+2a_{1121}x_1x_2+a_{1122}x_2^2+b_{11}\Big)dx_2=0,\\&\Big(a_{2211}x_1^2+2a_{2221}x_1x_2+a_{2222}x_2^2+b_{22}\Big)dx_1\\-&\Big(a_{2111}x_1^2+2a_{2121}x_1x_2+a_{2122}x_2^2+b_{21}\Big)dx_2=0\end{aligned} \tag{33}$$

on each smooth portion of ∂Q. Similar equations are derived from (24). Namely,

$$\begin{aligned}&\Big(a_{1121}y_1^2+2a_{2121}y_1y_2+a_{2221}y_2^2+c_{21}\Big)dy_1\\-&\Big(a_{1111}y_1^2+2a_{2111}y_1y_2+a_{2211}y_2^2+c_{11}\Big)dy_2=0,\\&\Big(a_{1122}y_1^2+2a_{2122}y_1y_2+a_{2222}y_2^2+c_{22}\Big)dy_1\\-&\Big(a_{1121}y_1^2+2a_{2121}y_1y_2+a_{2221}y_2^2+c_{21}\Big)dy_2=0\end{aligned} \tag{34}$$

on each smooth portion of ∂P.

When the forms (33) (34) are exact, then ∂Q and ∂P reduce to (arcs on) circles.

Proposition 2 *Suppose*

$$a_{1111}=-a_{2121}=a_{2222},\quad a_{2111}=-a_{2221},\quad a_{1121}=-a_{2122}, \tag{35}$$

and

$$a_{2111}=a_{1121}=0,\quad a_{1122}=a_{2211}=3a_{1111}, \tag{36}$$

$$b_{21}=c_{21}=0,\quad b_{11}=b_{22}=-\rho^2\,a_{1111},\quad c_{11}=c_{22}=-\sigma^2\,a_{1111} \tag{37}$$

where $\rho > 0$ and $\sigma > 0$. Then, we have

$$x_1^2 + x_2^2 = \rho^2 \quad \textit{and} \quad y_1^2 + y_2^2 = \sigma^2 \tag{38}$$

respectively on ∂Q and on ∂P.

Proof. (35) reduce (33) and (34) to the following equations:

$$\begin{cases} d\Big(a_{2111}\dfrac{x_1^3}{3} - a_{1122}\dfrac{x_2^3}{3} - a_{1111}x_1^2x_2 - a_{1121}x_1x_2^2 + b_{21}x_1 - b_{11}x_2\Big) = 0 \\ d\Big(a_{2211}\dfrac{x_1^3}{3} + a_{1121}\dfrac{x_2^3}{3} + a_{2111}x_1^2x_2 + a_{1111}x_1x_2^2 + b_{22}x_1 - b_{21}x_2\Big) = 0 \end{cases}$$

and

$$\begin{cases} d\Big(a_{1121}\dfrac{y_1^3}{3} - a_{2211}\dfrac{y_2^3}{3} - a_{1111}y_1^2y_2 - a_{2111}y_1y_2^2 + c_{21}y_1 - c_{11}y_2\Big) = 0 \\ d\Big(a_{1122}\dfrac{y_1^3}{3} + a_{2111}\dfrac{y_2^3}{3} + a_{1121}y_1^2y_2 + a_{1111}y_1y_2^2 + c_{22}y_1 - c_{21}y_2\Big) = 0 \end{cases}$$

respectively on ∂Q and on ∂P. It follows then

$$\Big(a_{2111}\frac{x_1^2}{3} - a_{1121}x_2^2 + b_{21}\Big)\, x_1 - \Big(a_{1122}\frac{x_2^2}{3} + a_{1111}x_1^2 + b_{11}\Big)\, x_2 = \mathrm{const}_1$$
$$\Big(a_{2211}\frac{x_1^2}{3} + a_{1111}x_2^2 + b_{22}\Big)\, x_1 + \Big(a_{2111}x_1^2 + a_{1121}\frac{x_2^2}{3} - b_{21}\Big)\, x_2 = \mathrm{const}_2$$

on ∂Q and similar equations involving y_1, y_2 on ∂P. This observation suggests that $\mathrm{const}_1 = \mathrm{const}_2 = 0$. In fact, (36) and (37) then yield to (38) as we have expected. Q.E.D.

Corollary 2 *(35) (36) (37) imply*

$$\begin{aligned} A(x,y) &= -\, x_1^2y_1^2 - 3x_1^2y_2^2 - 3x_2^2y_1^2 - x_2^2y_2^2 + 4x_1x_2y_1y_2, \\ b(y) &= \rho^2y_1^2 + \rho^2y_2^2, \quad c(x) = \sigma^2x_1^2 + \sigma^2x_2^2 \end{aligned} \tag{39}$$

when $a_{1111} = -1$. Hence, the operator $a(x,D)$ turns out

$$\begin{aligned} a(x,D)\cdot \;&= -\, D_1\Big((x_1^2 + 3x_2^2 - \rho^2)D_1\,\cdot\Big) - D_2\Big((x_2^2 + 3x_1^2 - \rho^2)D_2\,\cdot\Big) \\ &\quad + 2\, D_1\Big(x_1x_2\, D_2\,\cdot\Big) + 2\, D_2\Big(x_1x_2\, D_1\,\cdot\Big) \\ &\quad + \sigma^2\,(x_1^2 + x_2^2)\ \cdot\ . \end{aligned} \tag{40}$$

The left-hand sides of (31) and (32) become

$$(x_1^2 + x_2^2 - \rho^2)(3x_1^2 + 3x_2^2 - \rho^2) \quad \textit{and} \quad (y_1^2 + y_2^2 - \sigma^2)(3y_1^2 + 3y_2 - \sigma^2),$$

respectively.

Example 3 Recall (8) in Example 1. By an explicit computation, it is easy to verify the identity:

$$a(x,D)\,K_{Q,P}(x,x') - a(x',D')\,K_{Q,P}(x,x') = 0, \quad x,\, x' \in Q,$$

which constitutes the analogue to (4) in the present case.

Let us check the self-adjointness of the operator $a(x,D)$.

Proposition 3 *Let $Q = \{\, (x_1,x_2)\,;\, x_1^2+x_2^2 \le \rho^2 \,\}$. $a(x,D)$ is a self-adjoint operator in $\mathbf{L}^2(Q)$ with the domain $\mathbf{H}^2(Q)$, the Sobolev space of order 2 on Q.*

Before proceeding to the proof of the above proposition, we recall that the Sobolev space of order 2 on Q consisits of those functions $u(x_1,x_2)$, for which

$$u,\, \partial_j u,\, \partial_j\partial_k\, u \in \mathbf{L}^1(Q), \quad j=1,2,\, k=1,2,$$

where derivatives are taken in the weak sense. Since Q is a disk, it is convenient to give its representation in terms of the polar coordinates.

Lemma 7 *Suppose $u \in \mathbf{L}^2(Q)$. Let $U(r,\theta) = u(r\cos\theta, r\sin\theta)$, where $0<r<\rho$ and $0 \le \theta < 2\pi$. $u \in \mathbf{H}^2(Q)$ if and only if*

$$U,\, \frac{\partial}{\partial r}U,\, \frac{1}{r}\frac{\partial}{\partial\theta}U,\, \frac{\partial^2}{\partial r^2}U,\, \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}U + \frac{1}{r}\frac{\partial}{\partial r}U,\, \frac{1}{r^2}\frac{\partial}{\partial\theta}U - \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}U \tag{41}$$

are all in $\mathbf{L}^2((0,\rho)\times[0,2\pi), rdrd\theta)$.

In fact, we have

$$u^2 = U^2, \quad (\partial_1 u)^2 + (\partial_2 u)^2 = \Big(\frac{\partial}{\partial r}U\Big)^2 + \frac{1}{r^2}\Big(\frac{\partial}{\partial\theta}U\Big)^2,$$

and

$$\begin{aligned} &(\partial_1^2 u)^2 + 2(\partial_1\partial_2 u)^2 + (\partial_2^2 u)^2 \\ &= \Big(\frac{\partial^2}{\partial r^2}U\Big)^2 + \frac{2}{r^4}\Big(r\frac{\partial^2}{\partial r\partial\theta}U - \frac{\partial}{\partial\theta}U\Big)^2 + \frac{1}{r^4}\Big(\frac{\partial^2}{\partial\theta^2}U + r\frac{\partial}{\partial r}U\Big)^2 \end{aligned}$$

at $(x_1,x_2) = (r\cos\theta, r\sin\theta)$.

Now we proceed to the proof of Proposition 3. (40) in the polar coordinates turns out

$$\begin{aligned} A =& \frac{1}{r}\frac{\partial}{\partial r}\Big(r(r^2-\rho^2)\frac{\partial}{\partial r}\,\cdot\Big) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\Big((3r^2-\rho^2)\frac{\partial}{\partial\theta}\,\cdot\Big) + \sigma^2\, r^2\,\cdot \\ =& \Big\{ \frac{1}{r}\frac{\partial}{\partial r}\Big(r(r^2-\rho^2)\frac{\partial}{\partial r}\,\cdot\Big) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\Big((r^2-\rho^2)\frac{\partial}{\partial\theta}\,\cdot\Big) + \sigma^2\, r^2\,\cdot \Big\} + 2\frac{\partial^2}{\partial\theta^2}\,\cdot\;. \end{aligned}$$

Following[3] Reed and Simon ([4], p.160), we rewrite $\mathbf{L}^2(Q)$ as the tensor-product:

$$\mathbf{L}^2(Q) = \mathbf{L}^2((0,\rho), rdr) \otimes \mathbf{L}^2((0,2\pi], d\theta) = \oplus_{m=0}^{\infty} \mathbb{L}_m,$$

where $\mathbb{L}_m = \mathbf{L}^2((0,\rho), rdr) \otimes [\mathrm{e}^{\pm im\theta}]$. Actually, we see that this sum is an orthogonal direct sum. Therefore, for $U = \oplus_{m=0}^{\infty} U_m \in \oplus_{m=0}^{\infty} \mathbb{L}_m$, $U \in \mathbf{H}^2(Q)$ if and only if, for each m, $U_m(r,\theta) = U_{\pm m}(r)\,\mathrm{e}^{\pm im\theta} \in \mathbb{L}_m$ satisfies

$$U_{\pm m}(r),\ \frac{m^2}{r^2} U_{\pm m}(r),\ U'_{\pm m}(r),\ \frac{1}{r} U'_{\pm m}(r),\ U''_{\pm m}(r) \in \mathbf{L}^2((0,\rho], rdr) \qquad (42)$$

by (41), and their sums over m converge in $\mathbf{L}^2((0,\rho], rdr)$. Call $\mathbb{H}^2_m$ the corresponding subspace of such $U_m(r,\theta)$ in $\mathbb{L}_m$. Then on $\mathbb{H}^2_m$, the operator A reduces to

$$A_m = \left\{ \frac{1}{r} \frac{\partial}{\partial r} \Big(r(r^2 - \rho^2) \frac{\partial}{\partial r} \cdot \Big) + \sigma^2 r^2 \cdot \right\} - \Big(3 - \frac{\rho^2}{r^2} \Big) m^2 \ \cdot \ . \qquad (43)$$

It is clear that A_m with domain $\mathbb{H}^2_m$ is a closed symmetric operator and bounded from below by $-2m^2$. Thus, each A_m has a self-adjoint extension $\overline{A}_m$ in $\mathbb{L}_m$ with domain $\mathbb{D}_m \supset \mathbb{H}^2_m$. Then a self-adjoint extension of $a(x,D)$ is defined with the dmain $\mathbf{D}$, where

$$\mathbf{D} \ni u \quad \Longleftrightarrow \quad u = \oplus_{m=0}^{\infty} u_m, \quad u_m \in \mathbb{D}_m,$$

and $Au = \oplus_{m=1}^{\infty} \overline{A}_m\, u_m$ converges in $\mathbf{L}^2(Q)$. The proof of Proposition 3 is now complete by the following.

Lemma 8 *For each $m = 0, 1, 2, \cdots$, A_m is essentially self-adjoint in $\mathbb{L}_m$, and thus, $\mathbb{D}_m = \mathbb{H}^2_m$, $\overline{A}_m = A_m$.*

Proof. Consider $f(r,\theta) \in \mathbb{L}_m$ such that

$$\int_0^{\rho} \int_0^{2\pi} \Big\{ (A_m \pm i) u(r,\theta)\, \overline{f(r,\theta)} \Big\}\, d\theta\, rdr = 0 \qquad (44)$$

for any $u \in \mathbb{H}^2_m$. We will show $f(r,\theta) = 0$. It is then enough to take $f(r,\theta) = F_{\pm m}(r)\,\mathrm{e}^{\pm im\theta}$ and $u(r,\theta) = U_{\pm m}(r)\,\mathrm{e}^{\pm im\theta}$ for any $U_{\pm m}(r)$ which satisfies (42). Since the operator A_m is elliptic inside the interval $(0,\rho)$, we

[3] Suggested by Takashi Ichinose.

then immediately see $F_{\pm m}(r)$ is C^∞-smooth inside $(0,\rho)$. Thus, we consider the functional

$$\mathcal{F}_{\pm m}(U_{\pm m}) = \int_0^\rho U_{\pm m}(r)\,\overline{F_{\pm m}(r)}\,rdr$$
$$+\int_0^\rho \left\{(\rho^2-r^2)\,U'_{\pm m}(r)\,\overline{F'_{\pm m}(r)} + \rho^2\left(\frac{m}{r}U_{\pm m}(r)\right)\overline{\left(\frac{m}{r}F_{\pm m}(r)\right)}\right\} rdr$$

for smooth $U_{\pm m}(r)$ which vanish near $r=0$ and $r=\rho$. By (44), the functional $\mathcal{F}_{\pm m}(U_{\pm m})$ turns out

$$\int_0^\rho (3m^2-\sigma^2\,r^2 \mp i + 1)\,U_{\pm m}(r)\,\overline{F_{\pm m}(r)}\,rdr,$$

which is extended to a bounded linear functional on $\widehat{\mathbb{H}}^1_m$, where $\widehat{\mathbb{H}}^1_m$ is the Hilbert space consisting of $U(r)\in \mathbf{L}^2((0,\rho],rdr)$ such that

$$\frac{m}{r}\,U(r)\in \mathbf{L}^2((0,\rho],rdr), \quad \sqrt{\rho^2-r^2}\,U'(r)\in \mathbf{L}^2((0,\rho],rdr).$$

Here the derivative $U'(r)$ is in the weak sense. We take as the scalar product of $\widehat{\mathbb{H}}^1_m$, denoted by $\langle U,V\rangle$, the following integral:

$$\int_0^\rho \left\{(\rho^2-r^2)\,U'(r)\,\overline{V'(r)} + \rho^2\Big(\frac{m}{r}U(r)\Big)\overline{\Big(\frac{m}{r}V(r)\Big)} + U(r)\overline{V(r)}\right\} rdr$$

for U, $V\in\widehat{\mathbb{H}}^1_m$. Hence, by Riesz's representation theorem,$\mathcal{F}_{\pm m}(U_{\pm m}) = \langle U_{\pm m},\hat{F}_{\pm m}\rangle$ for some $\hat{F}_{\pm m}\in\mathbb{H}^1_m$. It is then not difficult to see that the above $\hat{F}_{\pm m}$ and $F_{\pm m}$ coincide, whence $F_{\pm m}\in\widehat{\mathbb{H}}^1_m$, and we take $U_{\pm m}=F_{\pm m}$. It then follows $F_{\pm m}=0$ in $\mathbf{L}^2((0,\rho],rdr)$. Q.E.D.

Remark 4.1 The ordinary differential operator A_m has three regular singularities at $r=0$, $r=\pm\rho$ and an irregular singularity of Poincaré rank 1 at $r=\infty$. In fact, A_m corresponds to the operator

$$r^2\,\frac{d^2}{dr^2}\;\cdot\; + p_1\Big(\frac{r}{\rho}\Big)\,r\,\frac{d}{dr}\;\cdot\; + p_2\Big(\frac{r}{\rho}\Big)\cdot\;,$$

where

$$p_1(t)=\frac{3t^2-1}{t^2-1}, \quad p_2(t)=\frac{\rho^2\sigma^2\,t^4}{t^2-1} - m^2\,p_1(t).$$

For $|t|>1$, note

$$p_1(t) = 1 + 2\sum_{k=0}^{\infty} t^{-2k}, \tag{45}$$

$$p_2(t) = t^2\left\{\rho^2\sigma^2 + (\rho^2\sigma^2-m^2)t^{-2} + (\rho^2\sigma^2-2m^2)\sum_{k=2}^{\infty} t^{-2k}\right\}. \tag{46}$$

An eigenfunction $u(r)$, which satisfies $A_m u = \lambda\, u$ in Q, can be written as $u(r) = U(\frac{r}{\rho})$. Here $U(t)$ is a solution of the equation

$$t^2 \frac{d^2}{dt^2} U(t) + P_1(t)\, t \frac{d}{dt} U(t) + \Big(P_2(t) - \frac{\lambda t^2}{t^2 - 1}\Big) U(t) = 0 \tag{47}$$

for $0 < t < 1$. We will discuss the solution of (47) elsewhere.

5 Some comments on the 3-dimensional case

Let $n = 3$. (23) in Proposition 1 implies that

$$a_{1k}(x)\, dx_2 \wedge dx_3 + a_{2k}(x)\, dx_3 \wedge dx_1 + a_{3k}(x)\, dx_1 \wedge dx_2 = 0 \tag{48}$$

for $k = 1, 2, 3$ at points on a smooth part of ∂Q. Here $a_{jk}(x)$ are those of (18). Similarly, (24) then reads as

$$\alpha_{1q}(y)\, dy_2 \wedge dy_3 + \alpha_{2q}(y)\, dy_3 \wedge dy_1 + \alpha_{3q}(y)\, dy_1 \wedge dy_2 = 0 \tag{49}$$

for $q = 1, 2, 3$ at points where ∂P is smooth. Here

$$\alpha_{pq}(y) = \sum_{j=1}^{3} \sum_{k=1}^{3} a_{jkpq} y_j y_k + c_{pq} \tag{50}$$

(See (22)).

In this situation, we have no chance to encounter with the case when both the surfaces ∂Q and ∂P are the spheres.

Proposition 4 *Suppose ∂Q and ∂P are the spheres centered at the origin, respectively of radius $\rho > 0$ and $\sigma > 0$. If (48) and (49) hold respectively on ∂Q and on ∂P, then all the a_{jkpq}, b_{jk} and c_{pq} vanish.*

Proof. Substitute $x_1 = \rho \sin\psi \cos\theta$, $x_2 = \rho \sin\psi \sin\theta$, $x_3 = \rho\cos\psi$ into (48) and $y_1 = \sigma \sin\psi' \cos\theta'$, $y_2 = \sigma \sin\psi' \sin\theta'$, $y_3 = \sigma\cos\psi'$ into (49). Rewrite the results into the forms of trigonometric polynomials. Since

$$\cos m\psi\, \cos n\theta,\ \cos m\psi\, \sin n\theta,\ \cdots$$

are linearly independent, their coefficients all vanish. Thus, we get 60 linear equations for 36 unknowns a_{jkpq}, b_{jk} and c_{pq} being taken as parameters. These linear equations are of rank 30. Soving them, we get $a_{jkpq} \equiv 0$ together with $b_{jk} = 0$ and $c_{pq} = 0$ as the compatibility requirements. We omit the details of computation, which can be verified with a use of any mathematics software. We, in fact, used the software Maple (of Waterloo, Inc.) Q.E.D.

We add some still further comments, alas rather of the nagative nature. All the forms (48) are closed if

$$\frac{\partial}{\partial x_1} a_{1k}(x) + \frac{\partial}{\partial x_2} a_{2k}(x) + \frac{\partial}{\partial x_3} a_{3k}(x) = 0, \quad k = 1, 2, 3.$$

This is the case if the coefficients a_{ijpq} satisfy the relations:

$$\begin{aligned}
&a_{1111} + a_{1212} + a_{1313} = 0, \quad && a_{1112} + a_{1222} + a_{1323} = 0, \\
&a_{1113} + a_{1223} + a_{1333} = 0, && a_{1211} + a_{2212} + a_{2313} = 0, \\
&a_{1212} + a_{2222} + a_{2323} = 0, && a_{1213} + a_{2223} + a_{2333} = 0, \\
&a_{1311} + a_{2312} + a_{3313} = 0, && a_{1312} + a_{2322} + a_{3323} = 0, \\
&a_{1313} + a_{2323} + a_{3333} = 0.
\end{aligned} \tag{51}$$

Actually, (51) also implies

$$\frac{\partial}{\partial y_1} \alpha_{1q}(y) + \frac{\partial}{\partial y_2} \alpha_{2q}(y) + \frac{\partial}{\partial y_3} \alpha_{3q}(y) = 0$$

for $q = 1, 2, 3$. Hence, all the forms (49) are then closed.

Lemma 9 *Suppose (51). Then all the forms (48) are exact.*

In fact, let

$$\begin{aligned}
g_{11}(x_1, x_2, x_3) =& a_{1223}\, x_2 x_3^2 - a_{1323}\, x_2^2 x_3 + \frac{1}{2} a_{1213}\, x_3^2 x_1 - \frac{1}{2} a_{1312}\, x_1 x_2^2 \\
& - \frac{1}{3} a_{1322}\, x_2^3 + \frac{1}{3} a_{1233}\, x_3^3 + b_{12}\, x_3, \\
g_{21}(x_1, x_2, x_3) =& a_{1313}\, x_3 x_1^2 - a_{1113}\, x_3^2 x_1 + \frac{1}{2} a_{1312}\, x_1^2 x_2 - \frac{1}{2} a_{1123}\, x_2 x_3^2 \\
& - \frac{1}{3} a_{1133}\, x_3^3 + \frac{1}{3} a_{1311}\, x_1^3 + b_{13}\, x_1, \\
g_{31}(x_1, x_2, x_3) =& a_{1112}\, x_1 x_2^2 - a_{1212}\, x_1^2 x_2 + \frac{1}{2} a_{1123}\, x_2^2 x_3 - \frac{1}{2} a_{1213}\, x_3 x_1^2 \\
& - \frac{1}{3} a_{1211}\, x_1^3 + \frac{1}{3} a_{1122}\, x_2^3 + b_{11}\, x_2.
\end{aligned} \tag{52}$$

Then

$$\begin{aligned}
& d\Big(g_{11}(x)\, dx_1 + g_{21}(x)\, dx_2 + g_{31}(x)\, dx_3\Big) \\
=& a_{11}(x)\, dx_2 \wedge dx_3 + a_{21}(x)\, dx_3 \wedge dx_1 + a_{31}(x)\, dx_1 \wedge dx_2.
\end{aligned}$$

Other $a_{jk}(x)$'s, $k = 2, 3$, $j = 1, 2, 3$, go similarly, and we obtain the corresponding $g_{ik}(x)$'s by cyclic permutations of indices, for which

$$\begin{aligned} &d\Big(g_{12}(x)\,dx_1 + g_{22}(x)\,dx_2 + g_{32}(x)\,dx_3\Big) \\ =&a_{12}(x)\,dx_2 \wedge dx_3 + a_{22}(x)\,dx_3 \wedge dx_1 + a_{32}(x)\,dx_1 \wedge dx_2 \end{aligned}$$

and

$$\begin{aligned} &d\Big(g_{13}(x)\,dx_1 + g_{23}(x)\,dx_2 + g_{33}(x)\,dx_3\Big) \\ =&a_{13}(x)\,dx_2 \wedge dx_3 + a_{23}(x)\,dx_3 \wedge dx_1 + a_{33}(x)\,dx_1 \wedge dx_2 \end{aligned}$$

hold. The lemma is thus proved.

Lemma 10 *Suppose (51). Then all the forms (49) are exact.*

In fact, this is verified in the same way as the previous lemma. Let

$$\begin{aligned} h_{11}(y) =&a_{2312}\,y_2y_3^2 - a_{2313}\,y_2^2y_3 + \frac{1}{2}a_{1312}\,y_3^2y_1 - \frac{1}{2}a_{1213}\,y_1y_2^2 \\ &- \frac{1}{3}a_{2213}\,y_2^3 + \frac{1}{3}a_{3313}\,y_3^2 + c_{12}\,y_3 \\ h_{21}(y) =&a_{1313}\,y_3y_1^2 - a_{1311}\,y_3^2y_1 + \frac{1}{2}a_{1213}\,y_1^2y_2 - \frac{1}{2}a_{2311}\,y_2y_3^2 \\ &- \frac{1}{3}a_{3311}\,y_3^2 + \frac{1}{3}a_{1113}\,y_1^3 + c_{13}\,y_1 \\ h_{31}(y) =&a_{1211}\,y_1y_2^2 - a_{1212}\,y_1^2y_2 + \frac{1}{2}a_{2311}\,y_2^2y_3 - \frac{1}{2}a_{1312}\,y_3y_1^2 \\ &- \frac{1}{3}a_{1112}\,y_1^3 + \frac{1}{3}a_{2211}\,y_2^3 + c_{11}\,y_2. \end{aligned} \tag{53}$$

Then

$$\begin{aligned} &d\Big(h_{11}(y)\,dy_1 + h_{21}(y)\,dy_2 + h_{31}(y)\,dy_3\Big) \\ =&\alpha_{11}(y)\,dy_2 \wedge dy_3 + \alpha_{21}(y)\,dy_3 \wedge dy_1 + \alpha_{31}(y)\,dy_1 \wedge dy_2. \end{aligned}$$

Other combinations of indices go similarly.

Remark 5.1 If

$$a_{jkpq} = a_{11pq}, \quad b_{jk} = b_{11}, \quad j,\,k = 1, 2, 3, \tag{54}$$

for all p, $q = 1, 2, 3$, then

$$\begin{aligned} a_{jk}(x) =&a_{11}(x) \\ =&a_{1111}\,x_1^2 + a_{1122}\,x_2^2 + a_{1133}\,x_3^2 \\ &+ 2a_{1112}\,x_1x_2 + 2a_{1123}\,x_2x_3 + 2a_{1113}\,x_1x_3 + b_{11} \end{aligned}$$

for all j, $k = 1, 2, 3$. Together with (48), we thus have

$$a_{11}(x) = -a_{1112}\,(x_1 - x_2)^2 - a_{1123}\,(x_2 - x_3)^2 - a_{1113}\,(x_3 - x_1)^2 + b_{11} \quad (55)$$

since then

$$a_{1111} + a_{1112} + a_{1113} = 0,\ a_{1113} + a_{1123} + a_{1133} = 0,$$
$$a_{1112} + a_{1122} + a_{1123} = 0.$$

Unfortunately we also have

$$\alpha_{pq}(y) = a_{11pq}\,(y_1 + y_2 + y_3)^2 + c_{pq} \quad (56)$$

for p, $q = 1, 2, 3$. Although ∂Q could nicely be defined, e.g., as an ellipsoid, by $a_{11}(x) = 0$, $\alpha_{pq}(y) = 0$ would not define any reasonable closed surface as ∂P.

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