On v-adic periods of t-motives

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May 31, 2011

Abstract

In this paper, we prove the equality between the transcendental degree of the field generated by the v-adic periods of a t-motive M and the dimension of the Tannakian Galois group for M, where v is a "finite" place of the rational function field over a finite field. As an application, we prove the algebraic independence of certain "formal" polylogarithms.

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1 Introduction

Let $\mathbb{F}_q$ be the finite field with q elements, θ and t be variables independent from each other, and $v \in \mathbb{F}_q[t]$ a fixed monic irreducible polynomial of degree d. Let M be a rigid analytically trivial t-motive over $\overline{\mathbb{F}_q(\theta)}$. Then there exists an ∞-adic period matrix for the Betti realization of M. Set Λ to be the field generated by the components of this matrix over $\overline{\mathbb{F}_q(\theta)}(t)$. Set Γ to be the Tannakian Galois group of M with respect to the Betti realization. Papanikolas [10] shows that the transcendental degree of Λ over $\overline{\mathbb{F}_q(\theta)}(t)$ coincides with the dimension of Γ. In this paper, we prove the v-adic analogue of this theorem.

Let $K/\mathbb{F}_q$ be a regular extension of fields. We set $K^{\mathrm{sep}}[t]_v := \varprojlim(K^{\mathrm{sep}}[t]/v^n)$ and $K^{\mathrm{sep}}(t)_v := \mathbb{F}_q(t) \otimes_{\mathbb{F}_q[t]} K^{\mathrm{sep}}[t]_v$, where K^{sep} is a separable closure of K. We also define $\mathbb{F}_q(t)_v$ and $K(t)_v$ by the same way. Let σ be the ring endomorphism $\sum a_i t^i \mapsto \sum a_i^q t^i$ of $K^{\mathrm{sep}}[t]$. Then σ naturally extends to an endomorphism of $K^{\mathrm{sep}}(t)_v$, also denoted by σ. A φ-module over $K(t)_v$ is a pair (M, φ) (or simply M) where M is a $K(t)_v$-vector space and $\varphi : M \to M$ is an additive map such that $\varphi(ax) = \sigma(a)\varphi(x)$ for all $a \in K(t)_v$ and $x \in M$. A morphism of φ-modules is a $K(t)_v$-linear map which is compatible with the φ's. A tensor product of two φ-modules is defined naturally.

For any φ-module M, we define the v-adic realization of M:

$$V(M) := (K^{\mathrm{sep}}(t)_v \otimes_{K(t)_v} M)^{\varphi},$$

where φ acts on $K^{\mathrm{sep}}(t)_v \otimes_{K(t)_v} M$ by $\sigma \otimes \varphi$ and $(-)^{\varphi}$ is the φ-fixed part. Then there exists a natural map

$$\iota_M : K^{\mathrm{sep}}(t)_v \otimes_{\mathbb{F}_q(t)_v} V(M) \to K^{\mathrm{sep}}(t)_v \otimes_{K(t)_v} M.$$

We can prove that ι_M is injective for each φ-module M. A φ-module M is said to be $K^{\mathrm{sep}}(t)_v$-trivial if M is finite-dimensional over $K(t)_v$ and ι_M is an isomorphism. Then the category of $K^{\mathrm{sep}}(t)_v$-trivial φ-modules over $K(t)_v$ equipped with the functor V forms a neutral Tannakian category over $\mathbb{F}_q(t)_v$. For any $K^{\mathrm{sep}}(t)_v$-trivial φ-module M, we denote by Γ_M the Tannakian Galois group of the Tannakian subcategory of $K^{\mathrm{sep}}(t)_v$-trivial φ-modules generated by M (see Subsections 3.2 and 3.3).

Let M be a finite-dimensional φ-module and $\mathbf{m} \in \mathrm{Mat}_{r\times 1}(M)$ a $K(t)_v$-basis of M. Then there exists a matrix $\Phi \in \mathrm{Mat}_{r\times r}(K(t)_v)$ such that $\varphi\mathbf{m} = \Phi\mathbf{m}$. If M is $K^{\mathrm{sep}}(t)_v$-trivial, we can take a matrix $\Psi = (\Psi_{ij})_{i,j} \in \mathrm{GL}_r(K^{\mathrm{sep}}(t)_v)$ such that $\Psi^{-1}\mathbf{m}$ forms an $\mathbb{F}_q(t)_v$-basis of $V(M)$. The entries of this matrix are called v-adic periods of M, which are our main objects of study in this paper. We set

$$\Sigma := K(t)_v[\Psi, 1/\det\Psi] := K(t)_v[\Psi_{11}, \Psi_{12}, \ldots, \Psi_{rr}, 1/\det\Psi] \subset K^{\mathrm{sep}}(t)_v.$$

Then Σ is stable under the σ-action. For any $\mathbb{F}_q(t)_v$-algebra R and S, we set $S^{(R)} := R \otimes_{\mathbb{F}_q(t)_v} S$. If σ acts on S, we define the σ-action on $S^{(R)}$ by $\mathrm{id} \otimes \sigma$. Set $\Gamma(R) := \mathrm{Aut}_{\sigma}(\Sigma^{(R)}/K(t)_v^{(R)})$ the group of automorphisms of $\Sigma^{(R)}$ over $K(t)_v^{(R)}$ that commute with σ. Then Γ forms a functor from the category of $\mathbb{F}_q(t)_v$-algebras to the category of groups. If we factorize $v = \prod_{l\in\mathbb{Z}/d}(t - \lambda_l)$ in $K^{\mathrm{sep}}[t]$ with $\lambda_l^q = \lambda_{l+1}$, then we can write $K^{\mathrm{sep}}(t)_v = \prod_l K^{\mathrm{sep}}((t - \lambda_l))$ and $\Psi_{ij} = (\Psi_{ijl})_l$ where $\Psi_{ijl} \in K^{\mathrm{sep}}((t - \lambda_l))$. We set

$$\Lambda_l := K(t)_v(\Psi_{11l}, \ldots, \Psi_{rrl}) \subset K^{\mathrm{sep}}((t - \lambda_l))$$

for each $l \in \mathbb{Z}/d$. Our main result in this paper is (see Lemma 4.16 and Theorems 4.14 and 5.15):

Theorem 1.1. *The functor Γ is representable by a smooth affine algebraic variety over $\mathbb{F}_q(t)_v$, also denoted by Γ. We have an equality $\dim \Gamma = \text{tr.deg}_{K(t)_v} \Lambda_l$ for each $l \in \mathbb{Z}/d$ and there exists a natural isomorphism $\Gamma \to \Gamma_M$ of affine group schemes over $\mathbb{F}_q(t)_v$.*

This theorem is a v-adic analogue of Papanikolas's Theorem 4.3.1 and 4.5.10 in [10], which treated ∞-adic objects. The proof of this theorem follows [10] closely, but since $K^{\text{sep}}(t)_v$ is not a field if $d > 1$, several arguments here are more complicated than in [10]. Let $K = \mathbb{F}_q(\theta)$ where θ is a variable independent of t. Papanikolas shows the equality of the transcendental degree of the field of periods (specialized at $t = \theta$) over K and the dimension of the Tannakian Galois group using the so-called ABP-criterion proved by Anderson, Brownawell and Papanikolas in [2]. In fact he proved an algebraic independence of Carlitz logarithms. On the other hand, Anderson and Thakur [3] shows that the relation between the Carlitz zeta values and Carlitz logarithms. Then using these results, Chang and Yu [5] determined the all algebraic relations among the Carlitz zeta values. These applications are our motivation of this paper, but in this paper, we can only prove a v-adic analogue of the ABP-criterion for the rank one case.

In Section 3, first we review a theory of φ-modules in a general setting and construct a Tannakian category. In the v-adic case, we show that this category is equivalent to the category of Galois representations. In Section 4, we consider Frobenius equations in our situation, and construct Γ. In Section 5, we discuss the relation between Γ and Γ_M, and prove that these are isomorphic in the v-adic case. This uses the fact that the set of $\mathbb{F}_q(t)_v$-valued points $\Gamma(\mathbb{F}_q(t)_v)$ is Zariski dense in Γ. Since $\Gamma(\mathbb{F}_q(t)_v)$ contains the Galois image, this is large enough in Γ. This is an essentially different point from Papanikolas's proof for the ∞-adic case, in which the Zariski density is not proved and other facts are used to show this isomorphism. In Section 6, we discuss a v-adic analogue of the ABP-criterion. In Section 7, we prove the algebraic independence of certain "formal" polylogarithms.

Acknowledgments. The author thanks Yuichiro Taguchi for many helpful discussions on the contents of this paper and for reading preliminary manuscripts of this paper carefully.

2 Notations and terminology

2.1 Table of symbols

$\mathbb{F}_q$	:=	the finite field of q elements
$\bar{k}$	:=	an algebraic closure of a field k
k^{sep}	:=	the separable closure of a field k in $\bar{k}$
$\#S$	:=	the cardinality of a set S
$\text{Mat}_{r \times s}(R)$	:=	the set of r by s matrices with entries in a ring or module R
$\text{GL}_r(R)$	:=	the group of invertible r by r matrices with entries in a ring R
$\mathbf{Vec}(k)$	:=	the category of finite-dimensional vector spaces over a field k
$\mathbf{Rep}(G, R)$	:=	for a ring R the category of finitely generated R-representations of an affine group scheme G over R, or for a topological ring R the category of finitely generated continuous R-representations of a topological group G

2.2 Action

Let R be a ring or module and $f : R \to R$ a map. For a matrix $A = (A_{ij})_{ij} \in \mathrm{Mat}_{r\times s}(R)$, we denote by $f(A)$ the matrix $(f(A_{ij}))_{ij}$.

Let S be a set and H a set of maps from S to itself. Then we denote by S^H the subset of S consisting of elements which are fixed by H. For a map $f : S \to S$, we set $S^f := S^{\{f\}}$.

2.3 Base change

Let $R \to S$ be a homomorphism of commutative rings and X a scheme over R. We denote by $X_S := X \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, S$ the base change from R to S of X. We also denote by $X(S) := \mathrm{Hom}_{\mathrm{Spec}\, R}(\mathrm{Spec}\, S, X)$ the set of S-valued points of X over R. When R and S are fields, we have a natural injection $X(R) \hookrightarrow X(S) \cong X_S(S)$. We always consider $X(R)$ as a subset of $X_S(S)$ via this injection.

3 φ-modules

3.1 étale φ-modules

In this subsection, we recall the definitions and properties of étale φ-modules (cf. [7]). Let A be a commutative ring and σ an endomorphism of A. For any A-module M, we put $M^{(\sigma)} := A \otimes_A M$, the scalar extension of M by σ. A map $\varphi : M \to M$ is said to be *σ-semilinear* if $\varphi(x + y) = \varphi(x) + \varphi(y)$ and $\varphi(ax) = \sigma(a)\varphi(x)$ for all $x, y \in M$ and $a \in A$. Then it is clear that to give a σ-semilinear map $\varphi : M \to M$ is equivalent to giving an A-linear map $\varphi_\sigma : M^{(\sigma)} \to M$.

Definition 3.1. A *φ-module* (M, φ) over (A, σ) (or simply, M over A) is an A-module M endowed with a σ-semilinear map $\varphi : M \to M$. A *morphism* of φ-modules is an A-linear map which is compatible with the φ's. When A is a noetherian ring, a φ-module (M, φ) is said to be *étale* if M is a finitely generated A-module and $\varphi_\sigma : M^{(\sigma)} \to M$ is bijective.

We denote by $\Phi\mathrm{M}_A$ the category of φ-modules over A and $\Phi\mathrm{M}_A^{\text{ét}}$ its full subcategory consisting of all étale φ-modules. For any φ-modules M and N, we denote by $\mathrm{Hom}_\varphi(M, N)$ the set of morphisms of M to N in $\Phi\mathrm{M}_A$.

Let $A_\sigma[\varphi]$ be the ring (non commutative if $\sigma \neq \mathrm{id}_A$) generated by A and an element φ with the relation

$$\varphi a = \sigma(a)\varphi$$

for each $a \in A$. Then it is clear that the category $\Phi\mathrm{M}_A$ and the category of $A_\sigma[\varphi]$-modules are naturally identified. Hence, the category $\Phi\mathrm{M}$ is an A^σ-linear abelian category.

For each φ-module M and N, we denote by $M \otimes N$ the *tensor product* of M and N, which is $M \otimes_A N$ as an A-module and has a φ-action defined by $\varphi \otimes \varphi$. Then the functor $\otimes$ is a bi-additive functor and (A, σ) is an identity object in $\Phi\mathrm{M}_A$ for this tensor product. Therefore the category $\Phi\mathrm{M}_A$ is an abelian tensor category ([6]).

Proposition 3.2. *There exists a natural isomorphism $A^\sigma \cong \mathrm{End}_\varphi(A) := \mathrm{Hom}_\varphi(A, A)$.*

Proof. For any endomorphism $f \in \mathrm{End}_\varphi(A)$, we have $\sigma(f(1)) = \varphi(f(1)) = f(\varphi(1)) = f(\sigma(1)) = f(1)$. Hence $f(1) \in A^\sigma$. Conversely for any element $a \in A^\sigma$, we have a map $f_a : A \to A; x \mapsto ax$. It is clear that $f_a \in \mathrm{End}_\varphi(A)$. These are inverse to each other. $\square$

Proposition 3.3. *Assume that A is noetherian and σ is flat. Then the category $\Phi\mathrm{M}_A^{\text{ét}}$ is an abelian A^σ-linear tensor category.*

Proof. It is clear that $\Phi\mathrm{M}_A^{\text{ét}}$ is closed under finite sums and tensor products, and the identity object (A, σ) is étale. Therefore it is enough to show that for each étale φ-modules M and N and a morphism $f : M \to N$, the kernel and cokernel of f in $\Phi\mathrm{M}_A$ are étale. Since M and N are étale and σ is flat, we have the commutative diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & (\ker f)^{(\sigma)} & \longrightarrow & M^{(\sigma)} & \longrightarrow & N^{(\sigma)} & \longrightarrow & (\operatorname{im} f)^{(\sigma)} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi'_\sigma} & & \downarrow{\scriptstyle \varphi_{M,\sigma}} & & \downarrow{\scriptstyle \varphi_{N,\sigma}} & & \downarrow{\scriptstyle \varphi''_\sigma} & & \\ 0 & \longrightarrow & \ker f & \longrightarrow & M & \longrightarrow & N & \longrightarrow & \operatorname{im} f & \longrightarrow & 0, \end{array}$$

where $\varphi_{M,\sigma}$ and $\varphi_{N,\sigma}$ are isomorphisms and the rows are exact. Then we have that φ'_σ and φ''_σ are isomorphism by a diagram chasing. □

Let (M, φ_M) and (N, φ_N) be φ-modules over A. If $\varphi_{M,\sigma} : M^{(\sigma)} \to M$ is an isomorphism, we define a φ-module $\mathrm{Hom}(M, N)$, whose underlying A-module is the space $\mathrm{Hom}_A(M, N)$ of A-module homomorphisms and a φ-action is defined by

$$\mathrm{Hom}_A(M, N)^{(\sigma)} \to \mathrm{Hom}_A(M^{(\sigma)}, N^{(\sigma)}) \to \mathrm{Hom}_A(M, N),$$

where the first map is the natural map and the second map is defined by $f \mapsto \varphi_{N,\sigma} \circ f \circ \varphi_{M,\sigma}^{-1}$. There exists a natural morphism of φ-modules $\mathrm{ev}_{M,N} : \mathrm{Hom}(M, N) \otimes M \to N$. For each M such that $\varphi_{M,\sigma}$ is an isomorphism, we set $M^\vee := \mathrm{Hom}(M, A)$ the dual of M.

Proposition 3.4. *Assume that A is noetherian and σ is flat. Then for any objects M and N in $\Phi\mathrm{M}_A^{\text{ét}}$, the φ-module* $\mathrm{Hom}(M, N)$ *is étale, the contravariant functor*

$$\Phi\mathrm{M}_A^{\text{ét}} \to \mathbf{Set};\ T \mapsto \mathrm{Hom}_\varphi(T \otimes M, N)$$

is representable by $\mathrm{Hom}(M, N)$ *and* $\mathrm{ev}_{M,N}$ *corresponds to* $\mathrm{id}_{\mathrm{Hom}(M,N)}$.

Proof. Since M and N are finitely generated and A is noetherian, $\mathrm{Hom}(M, N)$ is also finitely generated. Since σ is flat and M is finitely presented, the map $\mathrm{Hom}_A(M, N)^{(\sigma)} \to \mathrm{Hom}_A(M^{(\sigma)}, N^{(\sigma)})$ is an isomorphism ([4], Chap. I, Sect. 2, Prop. 11). Since $\varphi_{M,\sigma}$ and $\varphi_{N,\sigma}$ are bijective, the φ-module $\mathrm{Hom}(M, N)$ is étale. It is clear that there exists a natural isomorphism $\mathrm{Hom}_A(T \otimes M, N) \cong \mathrm{Hom}_A(T, \mathrm{Hom}(M, N))$ which is functorial in T. Then we can calculate that the subspaces $\mathrm{Hom}_\varphi(T \otimes M, N)$ and $\mathrm{Hom}_\varphi(T, \mathrm{Hom}(M, N))$ are corresponding with this isomorphism. The last assertion is clear. □

Proposition 3.5. *Assume that A is a field. Then the category $\Phi\mathrm{M}_A^{\text{ét}}$ is a rigid abelian A^σ-linear tensor category.*

Proof. By Proposition 3.3, $\Phi\mathrm{M}_A^{\text{ét}}$ is an abelian A^σ-linear tensor category. By Proposition 3.4, $\Phi\mathrm{M}_A^{\text{ét}}$ has internal homs. Therefore it is enough to show that the natural map

$$\otimes_{i\in I} \mathrm{Hom}(M_i, N_i) \to \mathrm{Hom}(\otimes_{i\in I} M_i, \otimes_{i\in I} N_i)$$

is an isomorphism for any finite families of objects $(M_i)_{i\in I}$ and $(Y_i)_{i\in I}$, and the natural map

$$M \to M^{\vee\vee}$$

is an isomorphism for any object M ([6]). These are true because A is a field. □

3.2 L-triviality

Let d be a positive integer and $F \subset E \subset L$ ring extensions where F, E are fields and $L = \prod_{l \in \mathbb{Z}/d} L_l$ is a finite product of fields. For each l, we sometimes consider L_l as a subset of L in an obvious way. Let $\sigma : L \to L$ be a ring endomorphism. We assume that the triple (F, E, L) satisfies the following properties:

- $\sigma(E) \subset E$ and $\sigma(L_l) \subset L_{l+1}$ for all l,
- $F = E^\sigma = L^\sigma$,
- L is a separable extension over E.

Such a triple (F, E, L) is called *σ-admissible*. See Lemma 3.24 for our main example. Another example can be found in [10].

Note that the separability of L over E is used to prove the smoothness of some algebraic groups (see Theorem 4.14), and not used in this section.

Remark 3.6. In [10], the term σ-admissible triple is defined only in the case where L is a field and σ is an isomorphism. Thus our general setting urges us to argue with greater care than in [10] at several points, and hence we decided not to avoid repeating similar arguments.

In this subsection, we consider φ-modules over $(E, \sigma|_E)$. For any φ-module M over E, we set

$$V(M) := (L \otimes_E M)^\varphi$$

where φ acts on $L \otimes_E M$ by $\sigma \otimes \varphi$. Then $V(M)$ is an F-vector space and V forms a functor. We have natural maps

$$\iota_M : L \otimes_F V(M) \to L \otimes_E M,$$

$$\iota_{M,l} : L_l \otimes_F V(M) \to L_l \otimes_E M \text{ for all } l.$$

Lemma 3.7. *Let M be a φ-module, and let $\mu_1, \dots, \mu_m \in V(M)$. If $\mu_1, \dots, \mu_m$ are linearly independent over F, then they are linearly independent over L (in $L \otimes_E M$).*

Proof. Assume that the lemma is not true. Then there exist $m \geq 1$, $\mu_1, \dots, \mu_m \in V(M)$ and $f_1, \dots, f_m \in L$ such that, $\mu_1, \dots, \mu_m$ are linearly independent over F, $(f_i)_i \neq 0$ and $\sum_i f_i \mu_i = 0$. We may assume that m is minimal among the integers which satisfy the above properties. We also assume that $f_1 = (a_l)_l \in \prod_l L_l$ is non-zero. Let $a_{l_0} \neq 0$. Then there exists an element $f' \in L$ such that $f'f = e_{l_0}$, where $e_{l_0} \in L$ is the element such that the l_0-th component is one and the other components are zero. Then we have $\sum_i f' f_i \mu_i = 0$. Therefore we may assume that $f_1 = e_{l_0}$. Then we have

$$0 = \sum_{j=0}^{d-1} \varphi^j (\sum_{i=1}^m f_i \mu_i) = \sum_{i=1}^m \sum_{j=0}^{d-1} \varphi^j (f_i \mu_i) = \sum_{i=1}^m (\sum_{j=0}^{d-1} \sigma^j (f_i)) \mu_i = \mu_1 + \sum_{i=2}^m (\sum_{j=0}^{d-1} \sigma^j (f_i)) \mu_i.$$

Therefore we may assume that $f_1 = 1$. Then we have

$$0 = \varphi(\sum_{i=1}^m f_i \mu_i) - \sum_{i=1}^m f_i \mu_i = \sum_{i=1}^m (\sigma(f_i) - f_i) \mu_i = \sum_{i=2}^m (\sigma(f_i) - f_i) \mu_i.$$

By the minimality of m, we have $f_i \in L^\sigma = F$ for all i. This contradicts the linear independence of $(\mu_i)_i$ over F. □

Corollary 3.8. *For any φ-module M, the maps ι_M and $\iota_{M,l}$ are injective and we have $\dim_F V(M) \leq \dim_E M$.*

Proof. By Lemma 3.7, ι_M is injective. It is clear that ι_M is injective if and only if $\iota_{M,l}$ are injective for all l. Therefore $\iota_{M,l}$ is injective and we have an inequality $\dim_F V(M) = \dim_{L_l}(L_l \otimes_F V(M)) \leq \dim_{L_l}(L_l \otimes_E M) = \dim_E M$. □

Definition 3.9. Let M be a finite-dimensional φ-module over E. We say that M is *L-trivial* if the map ι_M is an isomorphism.

We denote by $\Phi\mathrm{M}_E^L$ the full subcategory of $\Phi\mathrm{M}_E$ consisting of all L-trivial φ-modules. Let M be a finite-dimensional φ-module over E and $\mathbf{m} \in \mathrm{Mat}_{r\times 1}(M)$ its E-basis. Then there exists a matrix $\Phi \in \mathrm{Mat}_{r\times r}(E)$ such that $\varphi\mathbf{m} = \Phi\mathbf{m}$.

Proposition 3.10. *The following conditions are equivalent:*

(1) *M is L-trivial,*

(2) *$\iota_{M,l}$ is an isomorphism for each l,*

(3) *$\iota_{M,l}$ is an isomorphism for some l,*

(4) $\dim_F V(M) = \dim_E M$,

(5) *there exists a matrix $\Psi \in \mathrm{GL}_r(L)$ such that $\sigma\Psi = \Phi\Psi$.*

Proof. It is clear that (1) $\iff$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). Assume that the condition (4) is true. Then for each l, we have $\dim_{L_l}(L_l \otimes_F V(M)) = \dim_F V(M) = \dim_E M = \dim_{L_l}(L_l \otimes_E M)$. Therefore $\iota_{M,l}$ is an isomorphism. This means that the condition (4) implies the condition (2).

Assume that the condition (1) is true. Let $\mathbf{x}$ be an F-basis of $V(M)$. Since the natural map $\iota_M : L \otimes_F V(M) \to L \otimes_E M$ is an isomorphism, there exists a matrix $\Psi \in \mathrm{GL}_r(L)$ such that $\Psi\mathbf{x} = 1 \otimes \mathbf{m}$. Then we have

$$(\sigma\Psi)\mathbf{x} = (\sigma\Psi)(\varphi\mathbf{x}) = \varphi(\Psi\mathbf{x}) = \varphi(1 \otimes \mathbf{m}) = 1 \otimes \varphi\mathbf{m} = 1 \otimes \Phi\mathbf{m} = \Phi(1 \otimes \mathbf{m}) = \Phi\Psi\mathbf{x}.$$

By Lemma 3.7, we have $\sigma\Psi = \Phi\Psi$ and the condition (5) is true. Conversely, assume that the condition (5) is true. Then we have

$$\varphi(\Psi^{-1}(1 \otimes \mathbf{m})) = (\sigma\Psi)^{-1}(1 \otimes \varphi\mathbf{m}) = (\Phi\Psi)^{-1}(1 \otimes \Phi\mathbf{m}) = \Psi^{-1}(1 \otimes \mathbf{m}).$$

This means that $\Psi^{-1}(1 \otimes \mathbf{m}) \in \mathrm{Mat}_{r\times 1}(V(M))$. Thus we have an inequality $\dim_F V(M) \geq \dim_E M$ and the condition (4) is true. □

Corollary 3.11. *Let M be a finite-dimensional φ-module over E. If M is L-trivial then M is étale.*

Proof. By Proposition 3.10, there exists a matrix $\Psi \in \mathrm{GL}_r(L)$ such that $\sigma\Psi = \Phi\Psi$. Since σ is injective, we have $\det\Phi = \sigma(\det\Psi)\det\Psi^{-1} \neq 0$. □

Let M be an L-trivial φ-module over E, $\mathbf{m} \in \mathrm{Mat}_{r\times 1}(M)$ an E-basis of M and $\Phi \in \mathrm{GL}_r(E)$ a matrix such that $\varphi\mathbf{m} = \Phi\mathbf{m}$. By Proposition 3.10, there exists a matrix $\Psi \in \mathrm{GL}_r(L)$ such that $\sigma\Psi = \Phi\Psi$.

Definition 3.12. The matrix Ψ is called a *period matrix* of M in L or *fundamental matrix* of Φ, and the entries of Ψ are called *periods* of M in L.

Note that $\Psi' \in \mathrm{GL}_r(L)$ is another fundamental matrix of Φ if and only if $\Psi' = \Psi\delta$ for some $\delta \in \mathrm{GL}_r(F)$. Indeed, if $\sigma(\Psi') = \Phi\Psi'$ then $\sigma(\Psi^{-1}\Psi') = \sigma(\Psi)^{-1}\sigma(\Psi') = (\Phi\Psi)^{-1}(\Phi\Psi') = \Psi^{-1}\Psi'$, hence $\Psi^{-1}\Psi' \in \mathrm{GL}_r(L^\sigma) = \mathrm{GL}_r(F)$, and the converse is clear.

Proposition 3.13. *The period matrix Ψ of M is well-defined from M as an element of $\mathrm{GL}_r(E)\backslash \mathrm{GL}_r(L)/ \mathrm{GL}_r(F)$.*

Proof. Let $\mathbf{m}' \in \mathrm{Mat}_{r\times 1}(M)$ be another E-basis of M, $\Phi' \in \mathrm{GL}_r(E)$ and $\Psi' \in \mathrm{GL}_r(L)$ matrices such that $\varphi\mathbf{m}' = \Phi'\mathbf{m}'$ and $\sigma\Psi' = \Phi'\Psi'$. There exists a matrix $A \in \mathrm{GL}_r(E)$ which satisfies $\mathbf{m}' = A\mathbf{m}$. Then we have $\varphi\mathbf{m}' = \varphi(A\mathbf{m}) = \sigma(A)\varphi\mathbf{m} = \sigma(A)\Phi\mathbf{m} = \sigma(A)\Phi A^{-1}\mathbf{m}'$. Thus $\Phi' = \sigma(A)\Phi A^{-1}$. We also have $\sigma(A\Psi) = \sigma(A)\sigma(\Psi) = \sigma(A)\Phi\Psi = \Phi'(A\Psi)$. Hence we conclude that $\Psi' \in A\Psi \cdot \mathrm{GL}_r(F)$. □

Proposition 3.14. *The set of entries of $\Psi^{-1}(1 \otimes \mathbf{m})$ forms an F-basis of $V(M)$.*

Proof. By the proof of Proposition 3.10, we have that $\Psi^{-1}(1 \otimes \mathbf{m}) \in \mathrm{Mat}_{r\times 1}(V(M))$. Since $\dim_F V(M) = \dim_E M = r$, this is an F-basis of $V(M)$. □

Proposition 3.15. *The φ-module (E, σ) is L-trivial.*

Proof. We have equalities $V(E) = (L \otimes_E E)^\varphi = L^\sigma = F$. Therefore $\dim_F V(E) = 1 = \dim_E E$. □

Proposition 3.16. *Let M and N be L-trivial φ-modules. Then $M \oplus N$, $M \otimes N$ and $\mathrm{Hom}(M, N)$ are also L-trivial.*

Proof. Let $\mathbf{m} \in \mathrm{Mat}_{r\times 1}(E)$ be an E-basis of M, $\Phi_M \in \mathrm{GL}_r(E)$ the matrix such that $\varphi\mathbf{m} = \Phi_M\mathbf{m}$ and $\Psi_M \in \mathrm{GL}_r(L)$ a matrix which satisfies $\sigma\Psi_M = \Phi_M\Psi_M$. We also set $\mathbf{n} \in \mathrm{Mat}_{s\times 1}(E)$ an E-basis of N and $\Phi_N \in \mathrm{GL}_s(E), \Psi_N \in \mathrm{GL}_s(L)$ matrices which satisfy $\varphi\mathbf{n} = \Phi_N\mathbf{n}$ and $\sigma\Psi_N = \Phi_N\Psi_N$. We set

$$\mathbf{m} \oplus \mathbf{n} := \begin{bmatrix} \mathbf{m} \\ \mathbf{n} \end{bmatrix}, \Phi_M \oplus \Phi_N := \begin{bmatrix} \Phi_M & 0 \\ 0 & \Phi_N \end{bmatrix} \text{ and } \Psi_M \oplus \Psi_N := \begin{bmatrix} \Psi_M & 0 \\ 0 & \Psi_N \end{bmatrix}.$$

Then it is clear that $\mathbf{m} \oplus \mathbf{n}$ is an E-basis of $M \oplus N$, $\varphi(\mathbf{m} \oplus \mathbf{n}) = (\Phi_M \oplus \Phi_N)(\mathbf{m} \oplus \mathbf{n})$ and $\sigma(\Psi_M \oplus \Psi_N) = (\Phi_M \oplus \Phi_N)(\Psi_M \oplus \Psi_N)$. Therefore $M \oplus N$ is L-trivial.

Set $\mathbf{m} \otimes \mathbf{n}$ to be an E-basis of $M \otimes N$ naturally obtained from $\mathbf{m}$ and $\mathbf{n}$. Let $\Phi_M \otimes \Phi_N$ be the Kronecker product of Φ_M and Φ_N, and $\Psi_M \otimes \Psi_N$ be the Kronecker product of Ψ_M and Ψ_N. Then it is clear that $\varphi(\mathbf{m} \otimes \mathbf{n}) = (\Phi_M \otimes \Phi_N)(\mathbf{m} \otimes \mathbf{n})$ and $\sigma(\Psi_M \otimes \Psi_N) = (\Phi_M \otimes \Phi_N)(\Psi_M \otimes \Psi_N)$. Therefore $M \otimes N$ is L-trivial.

Let $\mathbf{m}^\vee$ be the dual basis of $\mathbf{m}$ for $M^\vee$. Then we have equalities $\varphi\mathbf{m}^\vee = (\Phi_M^{-1})^{\mathrm{tr}}\mathbf{m}^\vee$ and $\sigma(\Psi_M^{-1})^{\mathrm{tr}} = (\Phi_M^{-1})^{\mathrm{tr}}(\Psi_M^{-1})^{\mathrm{tr}}$, where A^{tr} is the transpose of a matrix A. Therefore $M^\vee$ is L-trivial. Since $\Phi\mathrm{M}_E^{\text{ét}}$ is a rigid tensor category, we have an isomorphism $M^\vee \otimes N \cong \mathrm{Hom}(M, N)$. Therefore $\mathrm{Hom}(M, N)$ is L-trivial. □

Proposition 3.17. *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence in $\Phi\mathrm{M}_E$. If M is L-trivial, then M' and M'' are also L-trivial.*

Proof. Let $\kappa : L \otimes_F V(M) \to L \otimes_F V(M'')$ be the natural map and $\iota'' : \mathrm{im}(\kappa) \to L \otimes_E M''$ be the restriction of the map $\iota_{M''}$. Then we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L \otimes_F V(M') & \longrightarrow & L \otimes_F V(M) & \longrightarrow & \mathrm{im}(\kappa) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \iota_{M'}} & & \downarrow{\scriptstyle \iota_M} & & \downarrow{\scriptstyle \iota''} & & \\ 0 & \longrightarrow & L \otimes_E M' & \longrightarrow & L \otimes_E M & \longrightarrow & L \otimes_E M'' & \longrightarrow & 0, \end{array}$$

where the rows are exact, ι_M is an isomorphism and $\iota_{M'}, \iota''$ are injective. Then we have that $\iota_{M'}$ and ι'' are isomorphism by a diagram chasing. Hence M' and M'' are L-trivial. □

Proposition 3.18. *The category $\Phi\mathrm{M}_E^L$ is a rigid abelian F-linear tensor category.*

Proof. By Proposition 3.5 and Corollary 3.11, it is enough to show that the category $\Phi\mathrm{M}_E^L$ is closed under direct sum, subquotient, tensor product and internal hom, and has an identity object for the tensor product. By Propositions 3.15, 3.16 and 3.17, these are true. □

By Corollary 3.8, we can consider V as a functor from $\Phi\mathrm{M}_E^L$ to the category of finite-dimensional F-vector spaces $\mathbf{Vec}(F)$.

Proposition 3.19. *The functor $V : \Phi\mathrm{M}_E^L \to \mathbf{Vec}(F)$ is F-linear and exact.*

Proof. It is clear that V is F-linear. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence in $\Phi\mathrm{M}_E^L$. It is clear that $0 \to V(M') \to V(M) \to V(M)$ is exact. This means that $\dim_F V(M) \leq \dim_F V(M') + \dim_F V(M'')$. On the other hand, we have $\dim_F V(M) = \dim_E M = \dim_E M' + \dim_E M'' \geq \dim_F V(M') + \dim_F V(M'')$. □

Proposition 3.20. *The functor $V : \Phi\mathrm{M}_E^L \to \mathbf{Vec}(F)$ is faithful.*

Proof. Let M and N be L-trivial φ-modules and $\phi : M \to N$ a morphism in $\Phi\mathrm{M}_E$. Then we have an exact sequence

$$0 \longrightarrow V(\ker \phi) \longrightarrow V(M) \xrightarrow{V(\phi)} V(N).$$

Therefore if $V(\phi) = 0$ then $V(\ker \phi) = V(M)$. Since M is L-trivial, we have an inequality $\dim_E \ker \phi \geq \dim_F V(\ker \phi) = \dim_F V(M) = \dim_E M$. This means that $\ker \phi = M$ and $\phi = 0$. □

Proposition 3.21. *Let M and N be L-trivial φ-modules. Then there exists a natural isomorphism $V(M) \otimes_F V(N) \to V(M \otimes N)$. The functor $V : \Phi\mathrm{M}_E^L \to \mathbf{Vec}(F)$ is a tensor functor with respect to this isomorphism.*

Proof. It is clear that there exists a natural isomorphism $(L \otimes_E M) \otimes_L (L \otimes_E N) \cong L \otimes_E (M \otimes N)$ which preserves φ-actions. By Lemma 3.7, the natural map $V(M) \otimes_F V(N) \to (L \otimes_E M) \otimes_L (L \otimes_E N)$ is injective. Therefore we have a natural injection

$$V(M) \otimes_F V(N) \hookrightarrow ((L \otimes_E M) \otimes_L (L \otimes_E N))^{\varphi} \cong (L \otimes_E (M \otimes N))^{\varphi} = V(M \otimes N).$$

Since $\dim_F(V(M) \otimes_F V(N)) = \dim_F V(M \otimes N)$, this map is a bijection. It is clear that this isomorphism is compatible with the associativity and the commutativity of tensor functors. It is also clear that $V(E) = F$. Thus the functor V is a tensor functor ([6], Definition 1.8). □

Recall that a *neutral Tannakian category* over a field k is a rigid abelian k-linear tensor category $\mathcal{C}$ for which $k \xrightarrow{\sim} \mathrm{End}(\mathbf{1})$ and there exists an exact faithful k-linear tensor functor $\omega : \mathcal{C} \to \mathbf{Vec}(k)$, where $\mathbf{1}$ is the unit object of $\mathcal{C}$ ([6], Definition 2.19). Any such functor ω is said to be a *fiber functor* for $\mathcal{C}$.

Theorem 3.22. *The category $\Phi\mathrm{M}_E^L$ is a neutral Tannakian category over F. The functor $V : \Phi\mathrm{M}_E^L \to \mathbf{Vec}(F)$ is a fiber functor for $\Phi\mathrm{M}_E^L$.*

Proof. By Proposition 3.2, we have $\mathrm{End}_\varphi(E) \cong E^\sigma = F$. By Proposition 3.18, the category $\Phi\mathrm{M}_E^L$ is a rigid abelian F-linear tensor category. By Propositions 3.19, 3.20 and 3.21, the functor $V : \Phi\mathrm{M}_E^L \to \mathbf{Vec}(F)$ is an exact faithful F-linear tensor functor. □

Let M be an L-trivial φ-module over E. We set $\mathcal{T}_M$ to be the Tannakian subcategory of $\Phi\mathrm{M}_E^L$ generated by M, and $V_M : \mathcal{T}_M \to \mathbf{Vec}(F)$ the restriction of the functor V. We denote by Γ_M the Tannakian Galois group of $(\mathcal{T}_M, V_M)$. For any F-algebra R, we define the functor $V_M^R : \mathcal{T}_M \to \mathbf{Mod}(R)$ by $N \mapsto R \otimes_F V(N)$, where $\mathbf{Mod}(R)$ is the category of finitely generated R-modules. Then by the definition of Γ_M, we have

$$\Gamma_M(R) = \mathrm{Aut}^{\otimes}(V_M^R)$$

where $\mathrm{Aut}^{\otimes}(V_M^R)$ is the group of invertible natural transformations from V_M^R to itself which is compatible with the tensor products. Therefore we have a canonical injection $\Gamma_M(R) \hookrightarrow \mathrm{GL}(R \otimes_F V(M))$.

3.3 v-adic case

Let t be a variable and $v \in \mathbb{F}_q[t]$ a fixed monic irreducible polynomial of degree d. For any field k containing $\mathbb{F}_q$, we set $k[t]_v := \varprojlim(k[t]/v^n)$ and $k(t)_v := \mathbb{F}_q(t) \otimes_{\mathbb{F}_q[t]} k[t]_v$.

Let σ be the ring endomorphism of $k[t]$

$$\sum a_i t^i \mapsto \sum a_i^q t^i.$$

Then σ naturally extends to an endomorphism of $k(t)_v$, also denoted by σ. Let k' be a splitting field of v over k in $\bar{k}$, and we factorize $v = \prod_{l \in \mathbb{Z}/d}(t - \lambda_l)$ in $k'[t]$ with $\lambda_l^q = \lambda_{l+1}$ for all $l \in \mathbb{Z}/d$. Then we have $k'(t)_v = \prod_{l \in \mathbb{Z}/d} k'((t - \lambda_l))$, and for any $a = (\sum_i a_{l,i}(t-\lambda_l)^i)_l \in k'(t)_v$,

$$\sigma(a) = (\sum a_{l-1,i}^q (t - \lambda_l)^i)_l.$$

Lemma 3.23. *For any field k containing $\mathbb{F}_q$, we have $(k(t)_v)^\sigma = \mathbb{F}_q(t)_v$.*

Proof. Clearly, $\mathbb{F}_q(t)_v = (\mathbb{F}_q(t)_v)^\sigma \subset (k(t)_v)^\sigma \subset (k'(t)_v)^\sigma$. By the explicit description of the σ-action as above, we have $(k'(t)_v)^\sigma = \{(\sum a_{l,i}(t-\lambda_l)^i)_l \in \mathbb{F}_{q^d}(t)_v | a_{l,i}^q = a_{l+1,i} \text{ for all } l \text{ and } i\}$. This set is isomorphic to $\mathbb{F}_{q^d}((t-\lambda_l))$ via the l-th projection for any l. On the other hand, we have $\mathbb{F}_q(t)_v \cong \mathbb{F}_{q^d}((t-\lambda_l))$. Thus the above inclusions are all equalities. □

Fix a field K containing $\mathbb{F}_q$ and assume that $K \cap \overline{\mathbb{F}_q} = \mathbb{F}_q$. Note that if $\mathbb{F}_q$ is not algebraically closed in K, then $K(t)_v$ may not be a field and the situation becomes more complicated. Thus in this paper, we always assume that $K \cap \overline{\mathbb{F}_q} = \mathbb{F}_q$.

Lemma 3.24. *The triple $(\mathbb{F}_q(t)_v, K(t)_v, K^{\mathrm{sep}}(t)_v)$ is σ-admissible.*

Proof. Since v is irreducible in $K[t]$, $K(t)_v$ is a field. By Lemma 3.23, we have $\mathbb{F}_q(t)_v = (K(t)_v)^\sigma = (K^{\mathrm{sep}}(t)_v)^\sigma$. We need to check the separability. Fix an l. We need to show that $K^{\mathrm{sep}}((t-\lambda_l))/K(t)_v$ is a separable field extension. It is clear that $K(t)_v = K'((t-\lambda_l))$ where $K' = K(\lambda_l)$. On the other hand, $K^{\mathrm{sep}}((t-\lambda_l))/K'((t-\lambda_l))$ is separable since K^{sep}/K' is separable ([9], Exercise 26.2). $\square$

Let $G_K := \mathrm{Gal}(K^{\mathrm{sep}}/K)$ be the absolute Galois group of K. Then G_K acts on $K^{\mathrm{sep}}[t]$ in an obvious way. This action naturally extends to an action on $K^{\mathrm{sep}}(t)_v$. For each $\tau \in G_K$ and $a = (\sum_i a_{l,i}(t-\lambda_l)^i)_l \in \prod_l K^{\mathrm{sep}}((t-\lambda_l))$, we have

$$\tau a = (\sum_i \tau a_{l+n,i}(t-\lambda_l)^i)_l,$$

where $n \in \mathbb{Z}/d$ is an element such that $\tau|_{\mathbb{F}_{q^d}} = \sigma|_{\mathbb{F}_{q^d}}^{-n}$. It is clear that this action is compatible with σ.

From now on, we consider φ-modules over the σ-admissible triple $(\mathbb{F}_q(t)_v,\ K(t)_v,\ K^{\mathrm{sep}}(t)_v)$. Let M be an étale φ-module over $K(t)_v$. The Galois group G_K acts on $K^{\mathrm{sep}}(t)_v \otimes M$ continuously by $\tau \otimes \mathrm{id}$ for each $\tau \in G_K$. Since this action is compatible with σ, the $\mathbb{F}_q(t)_v$-subspace $V(M)$ is G_K-stable. We denote by $V_K(M)$ this Galois representation. Conversely for any object V of $\mathbf{Rep}(G_K, \mathbb{F}_q(t)_v)$, we set

$$D(V) := (K^{\mathrm{sep}}(t)_v \otimes_{\mathbb{F}_q(t)_v} V)^{G_K},$$

where G_K acts on $K^{\mathrm{sep}}(t)_v \otimes_{\mathbb{F}_q(t)_v} V$ by $\tau \otimes \tau$ for $\tau \in G_K$. Then we can define a φ-action on $D(V)$ by $\sigma \otimes \mathrm{id}$.

Let M_0 be an étale φ-module over $K[t]_v$. Then we can define an $\mathbb{F}_q[t]_v$-representation of G_K

$$V_0(M_0) := (K^{\mathrm{sep}}[t]_v \otimes_{K[t]_v} M_0)^{\varphi},$$

where φ acts on $K^{\mathrm{sep}}[t]_v \otimes_{K[t]_v} M_0$ by $\sigma \otimes \varphi$ and G_K acts on $V_0(M_0)$ by $\tau \otimes \mathrm{id}$ for $\tau \in G_K$. Conversely for any object T of $\mathbf{Rep}(G_K, \mathbb{F}_q[t]_v)$, we set

$$D_0(T) := (K^{\mathrm{sep}}[t]_v \otimes_{\mathbb{F}_q[t]_v} T)^{G_K},$$

where G_K acts on $K^{\mathrm{sep}}[t]_v \otimes_{\mathbb{F}_q[t]_v} T$ by $\tau \otimes \tau$ for $\tau \in G_K$. Then we can define a φ-action on $D_0(T)$ by $\sigma \otimes \mathrm{id}$.

Theorem 3.25 ([8], Appendix)**.** (1) *For any étale φ-module M_0 over $K[t]_v$, the natural map*

$$K^{\mathrm{sep}}[t]_v \otimes_{\mathbb{F}_q[t]_v} V_0(M_0) \to K^{\mathrm{sep}}[t]_v \otimes_{K[t]_v} M_0$$

is bijective.

(2) *For any $\mathbb{F}_q[t]_v[G_K]$-module T of finite type over $\mathbb{F}_q[t]_v$, the natural map*

$$K^{\mathrm{sep}}[t]_v \otimes_{K[t]_v} D_0(T) \to K^{\mathrm{sep}}[t]_v \otimes_{\mathbb{F}_q[t]_v} T$$

is bijective and the φ-module $D_0(T)$ is étale.

(3) *The functor $V_0 : \Phi\mathrm{M}^{\text{ét}}_{K[t]_v} \to \mathbf{Rep}(G_K, \mathbb{F}_q[t]_v)$ is a tensor equivalence, with a quasi-inverse $D_0 : \mathbf{Rep}(G_K, \mathbb{F}_q[t]_v) \to \Phi\mathrm{M}^{\text{ét}}_{\mathbb{F}_q[t]_v}$.*

For any φ-module M_0 over $K[t]_v$, we can define a φ-action on $K(t)_v \otimes_{K[t]_v} M_0$ by $\sigma \otimes \varphi$.

Theorem 3.26. (1) *A φ-module M over $K(t)_v$ is $K^{\mathrm{sep}}(t)_v$-trivial if and only if there exists a subspace M_0 of M which is an étale φ-module over $K[t]_v$ such that $M = K(t)_v \otimes_{K[t]_v} M_0$.*

(2) *For any object V in $\mathbf{Rep}(G_K, \mathbb{F}_q(t)_v)$, the φ-module $D(V)$ is $K^{\mathrm{sep}}(t)_v$-trivial.*

(3) *The functor $V : \Phi\mathrm{M}_{K(t)_v}^{K^{\mathrm{sep}}(t)_v} \to \mathbf{Rep}(G_K, \mathbb{F}_q(t)_v)$ is a tensor equivalence, with a quasi-inverse $D : \mathbf{Rep}(G_K, \mathbb{F}_q(t)_v) \to \Phi\mathrm{M}_{K(t)_v}^{K^{\mathrm{sep}}(t)_v}$.*

Proof. Let M be a φ-module over $K(t)_v$ such that there exists a subspace M_0 which is an étale φ-module over $K[t]_v$ and $M = K(t)_v \otimes_{K[t]_v} M_0$. Then by Theorem 3.25 (1), we have an isomorphism $K^{\mathrm{sep}}[t]_v \otimes_{\mathbb{F}_q[t]_v} V_0(M_0) \cong K^{\mathrm{sep}}[t]_v \otimes_{K[t]_v} M_0$. By tensoring $K^{\mathrm{sep}}(t)_v$ to the both sides of this isomorphism, we conclude that M is $K^{\mathrm{sep}}(t)_v$-trivial.

Let V be an object in $\mathbf{Rep}(G_K, \mathbb{F}_q(t)_v)$. Then there exists a G_K-stable $\mathbb{F}_q[t]_v$-lattice T for V. It is clear that $D_0(T)$ is free over $K[t]_v$ and $D(V) = K(t)_v \otimes_{K[t]_v} D_0(T)$. Thus $D(V)$ is $K^{\mathrm{sep}}(t)_v$-trivial from the above argument and this proves (2). By Theorem 3.25 (2), we have an isomorphism $K^{\mathrm{sep}}(t)_v \otimes_{K(t)_v} D(V) \cong K^{\mathrm{sep}}(t)_v \otimes_{\mathbb{F}_q(t)_v} V$. By taking the φ-fixed parts of the both sides of this isomorphism, we have an isomorphism $V_K(D(V)) \cong V$.

Let M be a $K^{\mathrm{sep}}(t)_v$-trivial φ-module over $K(t)_v$. Then we have an isomorphism $K^{\mathrm{sep}}(t)_v \otimes_{\mathbb{F}_q(t)_v} V_K(M) \cong K^{\mathrm{sep}}(t)_v \otimes_{K(t)_v} M$. By taking the G_K-fixed parts of the both sides of this isomorphism, we have an isomorphism $D(V_K(M)) \cong M$, and this proves (3). Therefore M comes from étale φ-module over $K[t]_v$ and this proves (1). $\square$

4 Frobenius equations

Throughout this section, we fix a σ-admissible triple (F, E, L).

Example 4.1. The case $(F, E, L) = (\mathbb{F}_q(t)_v, K(t)_v, K^{\mathrm{sep}}(t)_v)$ is our main example of a σ-admissible triple, where the notation and the σ-action are as in Subsection 3.3.

Example 4.2. Let $(F, E, L) = (\mathbb{F}_q(t)_v, K^{\mathrm{rad}}(t)_v, \bar{K}(t)_v)$ where $K^{\mathrm{rad}} := \cup_n K^{1/q^n}$, the maximal radical extension of K in $\bar{K}$. The automorphism of $K^{\mathrm{rad}}(t)$

$$\sum_i a_i t^i \mapsto \sum_i a_i^{1/q} t^i$$

is naturally extends to an automorphism of $\bar{K}(t)_v$. We define σ to be this action. Then $(\mathbb{F}_q(t)_v, K^{\mathrm{rad}}(t)_v, \bar{K}(t)_v)$ is a σ-admissible triple. Note that, in this case we need to put $L_l = \bar{K}(\!(t - \lambda_{-l})\!)$. Note also that we do not use this type in this paper. However, the σ-action of this type is used in [10] and [5].

4.1 The group Γ

Let r be a positive integer. Fix matrices $\Phi = (\Phi_{ij}) \in \mathrm{GL}_r(E)$ and $\Psi = (\Psi_{ij})_{i,j} \in \mathrm{GL}_r(L)$ such that Ψ is a fundamental matrix for Φ. Thus we have an equation

$$\sigma(\Psi) = \Phi\Psi.$$

This means that the matrices Φ and Ψ come from an L-trivial φ-module over E. Since $L = \prod_l L_l$, we can write $\Psi_{ij} = (\Psi_{ijl})_l$ for each i and j. We set $\Psi_l := (\Psi_{ijl})_{i,j} \in \mathrm{GL}_r(L_l)$. Then we have $\sigma(\Psi_l) = \Phi\Psi_{l+1}$ for all l.

Let $X := (X_{ij})$ be an $r \times r$ matrix of independent variables X_{ij}, and set $\Delta := \det(X)$. We set $E[X, \Delta^{-1}] := E[X_{11}, X_{12}, \ldots, X_{rr}, \Delta^{-1}]$. Similarly $E[\Psi, \Delta(\Psi)^{-1}]$ and $E[\Psi_l, \Delta(\Psi_l)^{-1}]$ are defined. We define E-algebra homomorphisms $\nu : E[X, \Delta^{-1}] \to L;\ X_{ij} \mapsto \Psi_{ij}$ and $\nu_l : E[X, \Delta^{-1}] \to L_l;\ X_{ij} \mapsto \Psi_{ijl}$. Set $\mathfrak{p} := \ker \nu$, $\Sigma := E[\Psi, \Delta(\Psi)^{-1}] \cong E[X, \Delta^{-1}]/\mathfrak{p}$, $Z := \operatorname{Spec} \Sigma$, $\mathfrak{p}_l := \ker \nu_l$, $\Sigma_l := E[\Psi_l, \Delta(\Psi_l)^{-1}] \cong E[X, \Delta^{-1}]/\mathfrak{p}_l$ and $Z_l := \operatorname{Spec} \Sigma_l$. Then Z_l are closed subschemes of Z and $Z = \cup_l Z_l$. Let $\Lambda := \operatorname{Frac}(\Sigma)$ and $\Lambda_l := \operatorname{Frac}(\Sigma_l)$, the total rings of fractions.

Set $\Psi_1 := (\Psi_{ij} \otimes 1)_{i,j}$, $\Psi_2 := (1 \otimes \Psi_{ij})_{i,j}$ and $\widetilde{\Psi} = (\widetilde{\Psi}_{ij})_{i,j} := \Psi_1^{-1}\Psi_2$ in $\mathrm{GL}_r(L \otimes_E L)$. Since $L \otimes_E L = \prod_{l,m} L_l \otimes_E L_m$, we can write $\widetilde{\Psi}_{ij} = (\widetilde{\Psi}_{ijlm})_{l,m}$ with $\widetilde{\Psi}_{ijlm} \in L_l \otimes_E L_m$ for each i and j. We define F-algebra homomorphisms $\mu : F[X, \Delta^{-1}] \to L \otimes_E L;\ X_{ij} \mapsto \widetilde{\Psi}_{ij}$ and $\mu_{lm} : F[X, \Delta^{-1}] \to L_l \otimes_E L_m;\ X_{ij} \mapsto \widetilde{\Psi}_{ijlm}$. Set $\mathfrak{q} := \ker \mu$, $\Gamma := \operatorname{Spec} F[X, \Delta^{-1}]/\mathfrak{q}$, $\mathfrak{q}_{lm} := \ker \mu_{lm}$ and $\Gamma_{lm} := \operatorname{Spec} F[X, \Delta^{-1}]/\mathfrak{q}_{lm}$. Then Γ_{lm} are closed subschemes of Γ and $\Gamma = \cup_{l,m} \Gamma_{lm}$. By the next lemma, we can set $\mathfrak{q}_m := \mathfrak{q}_{0,m} = \mathfrak{q}_{1,m+1} = \cdots$ and $\Gamma_m := \Gamma_{0,m} = \Gamma_{1,m+1} = \cdots$.

Lemma 4.3. *For any $l, m \in \mathbb{Z}/d$, we have $\mathfrak{q}_{lm} = \mathfrak{q}_{l+1,m+1} = \mathfrak{q}_{l+2,m+2} = \cdots$.*

Proof. Let $\tilde{L}$ be the inductive limit of the inductive system $L \to L \to L \to \cdots$, where the transition maps are σ. Then L is a subring of $\tilde{L}$ and σ is naturally extends to an automorphism of $\tilde{L}$. We can define a σ-action on $\tilde{L} \otimes_E \tilde{L}$ by $\sigma \otimes \sigma$. This is an isomorphism and $L \otimes_E L$ is stable under this action. Thus we obtain an injective endomorphism σ of $L \otimes_E L$. It is clear that $\sigma(L_l \otimes_E L_m) \subset L_{l+1} \otimes_E L_{m+1}$.

Write $\Psi_1 = (\Psi_{1,lm})_{l,m}$ and $\Psi_2 = (\Psi_{2,lm})_{l,m}$ with $\Psi_{i,lm} \in \mathrm{GL}_r(L_l \otimes_E L_m)$, and set $\widetilde{\Psi}^{(lm)} := (\widetilde{\Psi}_{ijlm})_{i,j} \in \mathrm{GL}_r(L_l \otimes_E L_m)$ for each l and m. Then we obtain the equality $\sigma(\widetilde{\Psi}^{(lm)}) = \sigma(\Psi_{1,lm})^{-1}\sigma(\Psi_{2,lm}) = (\Phi\Psi_{1,l+1,m+1})^{-1}(\Phi\Psi_{2,l+1,m+1}) = \widetilde{\Psi}^{(l+1,m+1)}$. For any $h(X) \in F[X, \Delta^{-1}]$, we have $h(\widetilde{\Psi}^{(lm)}) = 0$ if and only if $h(\widetilde{\Psi}^{(l+1,m+1)}) = 0$ since $\sigma(h(\widetilde{\Psi}^{(lm)})) = h(\widetilde{\Psi}^{(l+1,m+1)})$ and σ is injective on $L \otimes_E L$. This proves the lemma. □

For any $h(X) \in L[X, \Delta^{-1}]$, we denote by $h^\sigma(X)$ the polynomial obtained by applying σ to the coefficients of $h(X)$. We define two endomorphisms

$$\sigma_0 : L[X, \Delta^{-1}] \to L[X, \Delta^{-1}];\ h(X) \mapsto h^\sigma(X),$$

$$\sigma_1 : L[X, \Delta^{-1}] \to L[X, \Delta^{-1}];\ h(X) \mapsto h^\sigma(\Phi X).$$

Then $\sigma_0(L_l[X, \Delta^{-1}]) \subset L_{l+1}[X, \Delta^{-1}]$ and $\sigma_1(L_l[X, \Delta^{-1}]) \subset L_{l+1}[X, \Delta^{-1}]$.

Lemma 4.4. *We have $\sigma_1\mathfrak{p} \subset \mathfrak{p}$, $\sigma_1\mathfrak{p}_l \subset \mathfrak{p}_{l+1}$, $\sigma_0\mathfrak{q} = \mathfrak{q}$, $\sigma_0\mathfrak{q}_m = \mathfrak{q}_m$, $\sigma\nu = \nu\sigma_1|_{E[X,\Delta^{-1}]}$ and $\sigma\nu_l = \nu_{l+1}\sigma_1|_{E[X,\Delta^{-1}]}$ for each l and m.*

Proof. For any $h(X) \in E[X, \Delta^{-1}]$, we have $\nu_{l+1}(\sigma_1(h(X))) = (\sigma_1 h)(\Psi_{l+1}) = h^\sigma(\Phi\Psi_{l+1}) = h^\sigma(\sigma\Psi_l) = \sigma(h(\Psi_l)) = \sigma(\nu_l(h(X)))$. If $h \in \mathfrak{p}_l$, then $(\sigma_1 h)(\Psi_{l+1}) = \sigma(h(\Psi_l)) = 0$, and hence $\sigma_1 h \in \mathfrak{p}_{l+1}$. Since $\mathfrak{q}_l \subset F[X, \Delta^{-1}]$ and $\sigma_0|_{F[X,\Delta^{-1}]} = \mathrm{id}$, we have $\sigma_0\mathfrak{q}_l = \mathfrak{q}_l$. The other assertions are proved similarly. □

For any ring homomorphism $R \to S$ and any ideal $\mathfrak{a} \subset R[X, \Delta^{-1}]$, we set $\mathfrak{a}_S := \mathfrak{a} \cdot S[X, \Delta^{-1}]$, the extension ideal of $\mathfrak{a}$.

Lemma 4.5. *There exists a bijection between the set of ideals of $F[X, \Delta^{-1}]$ and the set of ideals of $L[X, \Delta^{-1}]$ which are σ_0-stable, via the extension and the restriction of ideals.*

Proof. For any ideal $\mathfrak{a} \subset F[X, \Delta^{-1}]$, it is clear that $\sigma_0 \mathfrak{a}_L \subset \mathfrak{a}_L$. Because of the faithfully flatness of the inclusion $F[X, \Delta^{-1}] \hookrightarrow L[X, \Delta^{-1}]$, we have $\mathfrak{a} = \mathfrak{a}_L \cap F[X, \Delta^{-1}]$.

Conversely, we take any ideal $\mathfrak{b} \subset L[X, \Delta^{-1}]$ with $\sigma_0 \mathfrak{b} \subset \mathfrak{b}$, and set $\mathfrak{a} := \mathfrak{b} \cap F[X, \Delta^{-1}]$. It is clear that $\mathfrak{b} \supset \mathfrak{a}_L$; thus we need to show that the converse inclusion $\mathfrak{b} \subset \mathfrak{a}_L$.

Take an F-basis $(g_i)_{i \in I}$ of $F[X, \Delta^{-1}]$. Then this is an L-basis of $L[X, \Delta^{-1}]$. For each $h = \sum_i b_i g_i \in L[X, \Delta^{-1}]$, we set $\mathrm{supp}(h) := \{i \in I | b_i \neq 0\}$ and $l(h) := \# \mathrm{supp}(h)$. We take $h \in \mathfrak{b}$ and show that $h \in \mathfrak{a}_L$ by induction on $l(h)$. If $l(h) = 0$, then $h = 0 \in \mathfrak{a}_L$. Now suppose that $l(h) > 0$, and assume that if $\tilde{h} \in \mathfrak{b}$ and $l(\tilde{h}) < l(h)$ then $\tilde{h} \in \mathfrak{a}_L$. Let $e_l \in L$ be the element such that the l-th component is one and the other components are all zero. Then it is clear that $\sigma e_l = e_{l+1}$. We write $h = \sum_i b_i g_i$ and take i_1 such that $b_{i_1} \neq 0$. Take l_0 such that the l_0-th component of b_{i_1} is non-zero. Then there exists an element $b' \in L$ such that $b' b_{i_1} = e_{l_0}$. Since $\mathfrak{b}$ is an ideal and σ_0-stable, we have

$$\mathfrak{b} \ni \sum_{j=0}^{d-1} \sigma_0^j (b'h) = \sum_{j=0}^{d-1} \sigma_0^j (b' \sum_i b_i g_i) = \sum_i \sum_{j=0}^{d-1} \sigma^j (b' b_i) g_i =: \sum_i c_i g_i =: h',$$

$c_{i_1} = \sum_j \sigma^j (b' b_{i_1}) = 1$ and $\mathrm{supp}(h') \subset \mathrm{supp}(h)$. Therefore $h - b_{i_1} h' \in \mathfrak{b}$ and $l(h - b_{i_1} h') < l(h)$. By induction hypothesis, we have $h - b_{i_1} h' \in \mathfrak{a}_L$. Hence it is enough to show that $h' \in \mathfrak{a}_L$. If $c_i \in F$ for all i, then $h' \in \mathfrak{b} \cap F[X, \Delta^{-1}] = \mathfrak{a} \subset \mathfrak{a}_L$. If $\# L_l \leq 3$ for some (hence for all) l, we can write the σ action on L by $(x_l)_l \mapsto (x_{l-1})_l$. Hence $c_i \in F$ for all i. Thus we assume that $c_{i_2} \in L \setminus F$ for some i_2 and $\# L_l \geq 4$ for all l.

We claim that, we can construct an element $\bar{h} = \sum_i a_i g_i \in \mathfrak{a}_L$ which has the properties that $\mathrm{supp}(\bar{h}) \subset \mathrm{supp}(h')$ and $a_{i_1} = 1$. We first show that the claim implies $h' \in \mathfrak{a}_L$. Since $h' - \bar{h} \in \mathfrak{b}$ and $l(h' - \bar{h}) < l(h') \leq l(h)$, we have $h' - \bar{h} \in \mathfrak{a}_L$ by induction hypothesis. Thus we have $h' \in \mathfrak{a}_L$ since $\bar{h} \in \mathfrak{a}_L$.

Now we prove the claim. First, we construct an element $\bar{h} = \sum_i a_i g_i \in \mathfrak{b}$ which has the properties that $\mathrm{supp}(\bar{h}) \subset \mathrm{supp}(h')$, $a_{i_1} = 1$, $a_{i_2} \in L^\times$ and $\sigma(a_{i_2}^{-1}) - a_{i_2}^{-1} \in L^\times$. If $d = 1$, then we can take $\bar{h} = h'$ since L is a field and $\sigma(c_{i_2}^{-1}) - c_{i_2}^{-1} \neq 0$. Thus we suppose that $d \geq 2$. Since $c_{i_2} = (c_{i_2,l})_l \notin F$, there exists an l_0 such that $\sigma c_{i_2, l_0 - 1} \neq c_{i_2, l_0}$. Thus there is an element $c' \in L$ such that $c'(\sigma c_{i_2} - c_{i_2}) = e_{l_0}$. We set

$$h'' := \sum_i \alpha_i g_i := \sum_i \sum_{j=0}^{d-1} \sigma_0^j (c'(\sigma c_i - c_i)) g_i = \sum_{j=0}^{d-1} \sigma_0^j (c'(\sigma_0 h' - h')) \in \mathfrak{b}.$$

Then we have $\alpha_{i_1} = 0$, $\alpha_{i_2} = 1$ and $\mathrm{supp}(h'') \subset \mathrm{supp}(h')$. For $f = (f_l)_l \in L$, consider the element $\bar{h} := h' - f h'' = g_{i_1} + (c_{i_2,l} - f_l)_l g_{i_2} + \cdots \in \mathfrak{b}$. For any $x = (x_l)_l \in L^\times$, $\sigma x^{-1} - x^{-1} \in L^\times$ if and only if $\sigma x_l \neq x_{l+1}$ for all l. Therefore, it is enough to take an element f such that $c_{i_2,l} \neq f_l$ and $\sigma(c_{i_2,l-1} - f_{l-1}) \neq c_{i_2,l} - f_l$ for all l. Since $\# L_l \geq 4$, we can take $(f_l)_l$ inductively so that $f_1 \in L_1 \setminus \{c_{i_2,1}\}$, $f_l \in L_l \setminus \{c_{i_2,l}, c_{i_2,l} - \sigma(c_{i_2,l-1} - f_{l-1})\}$ for $2 \leq l < d$ and $f_d \in L_d \setminus (\{c_{i_2,d}, c_{i_2,d} - \sigma(c_{i_2,d-1} - f_{d-1})\} \cup \sigma^{-1}(\sigma(c_{i_2,d}) - c_{i_2,1} + f_1))$. Then such $(f_l)_l$ satisfies the above properties. Next, we show that $\bar{h} \in \mathfrak{a}_L$. Since $\sigma_0 \bar{h} - \bar{h} \in \mathfrak{b}$ and $l(\sigma_0 \bar{h} - \bar{h}) < l(\bar{h}) \leq l(h)$, we have $\sigma_0 \bar{h} - \bar{h} \in \mathfrak{a}_L$ by induction hypothesis. Similarly, we can show that $\sigma_0(a_{i_2}^{-1} \bar{h}) - a_{i_2}^{-1} \bar{h} \in \mathfrak{a}_L$. Therefore we have $(\sigma(a_{i_2}^{-1}) - a_{i_2}^{-1}) \bar{h} = (\sigma_0(a_{i_2}^{-1} \bar{h}) - a_{i_2}^{-1} \bar{h}) - \sigma(a_{i_2}^{-1})(\sigma_0 \bar{h} - \bar{h}) \in \mathfrak{a}_L$. Since $(\sigma(a_{i_2}^{-1}) - a_{i_2}^{-1}) \in L^\times$, we have $\bar{h} \in \mathfrak{a}_L$. □

Lemma 4.6. *The map $\prod_l \mathfrak{b}_l \mapsto (\mathfrak{b}_l)_l$ is a bijection between the set of ideals of $L[X, \Delta^{-1}]$ which are σ_0-stable, and the set of families $(\mathfrak{b}_l)_l$ where $\mathfrak{b}_l$ is an ideal of $L_l[X, \Delta^{-1}]$ and $\sigma_0 \mathfrak{b}_l \subset \mathfrak{b}_{l+1}$ for all l.*

Proof. This is clear. □

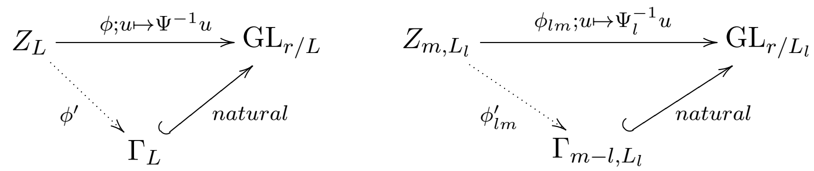
Lemma 4.7. *For each l, we give an ideal $\mathfrak{b}_l \subset L_l[X, \Delta^{-1}]$ such that $\sigma_0 \mathfrak{b}_l \subset \mathfrak{b}_{l+1}$. Then the restriction $\mathfrak{b}_l \cap F[X, \Delta^{-1}]$ is independent of l. The same is also true if we replace L_l by Σ_l.*

Proof. We only prove the case of L_l. Let ι be the natural injection $F[X, \Delta^{-1}] \hookrightarrow L[X, \Delta^{-1}]$ and π_l the natural projection $L[X, \Delta^{-1}] \twoheadrightarrow L_l[X, \Delta^{-1}]$ for each l. For each l, σ_0 induces a morphism $L_l[X, \Delta^{-1}] \to L_{l+1}[X, \Delta^{-1}]$; we also denote this by σ_0. Then we have an equality $\pi_{l+1}\iota = \sigma_0 \pi_l \iota$. For any $h \in \mathfrak{b}_l \cap F[X, \Delta^{-1}] = (\pi_l \iota)^{-1}\mathfrak{b}_l$, we have $\pi_{l+1}(\iota(h)) = \sigma_0(\pi_l(\iota(h))) \in \sigma_0 \mathfrak{b}_l \subset \mathfrak{b}_{l+1}$. Hence $h \in (\pi_{l+1}\iota)^{-1}\mathfrak{b}_{l+1} = \mathfrak{b}_{l+1} \cap F[X, \Delta^{-1}]$. Therefore we obtain $\mathfrak{b}_l \cap F[X, \Delta^{-1}] \subset \mathfrak{b}_{l+1} \cap F[X, \Delta^{-1}]$. Since the index set $\mathbb{Z}/d$ is a finite cyclic group, this inclusion is an equality. □

For any ring R, we denote by $\mathrm{GL}_{r/R}$ the R-group scheme of $r \times r$ invertible matrices.

Proposition 4.8. (1) *Let $\phi : Z_L \to \mathrm{GL}_{r/L}$ be the morphism of affine L-schemes defined by $u \mapsto \Psi^{-1}u$ for any L-algebra S and any S-valued point $u \in Z(S)$. Then ϕ factors through an isomorphism $\phi' : Z_L \to \Gamma_L$ of affine L-schemes.*

(2) *For any l and m, let $\phi_{lm} : Z_{m,L_l} \to \mathrm{GL}_{r/L_l}$ be the morphism of affine L_l-schemes defined by $u \mapsto \Psi_l^{-1}u$ for any L_l-algebra S and any S-valued point $u \in Z_m(S)$. Then ϕ_{lm} factors through an isomorphism $\phi'_{lm} : Z_{m,L_l} \to \Gamma_{m-l,L_l}$ of affine L_l-schemes.*



Proof. We prove only (2). Then (1) can be proved by the same argument. We define two L_l-algebra homomorphisms:

$$\alpha_l : L_l[X, \Delta^{-1}] \to L_l[X, \Delta^{-1}];\ X \mapsto \Psi_l^{-1}X, \tag{4.1}$$

$$\bar{\alpha}_{lm} : L_l[X, \Delta^{-1}] \xrightarrow{\alpha_l} L_l[X, \Delta^{-1}] \twoheadrightarrow L_l[X, \Delta^{-1}]/\mathfrak{p}_{m,L_l} = L_l \otimes_E E[X, \Delta^{-1}]/\mathfrak{p}_m. \tag{4.2}$$

Then ϕ_{lm} corresponds to $\bar{\alpha}_{lm}$ on the level of coordinate rings. Thus it is enough to show that $\alpha_l^{-1}\mathfrak{p}_{m,L_l} = \mathfrak{q}_{m-l,L_l}$.

For any $h(X) \in L_l[X, \Delta^{-1}]$, we have

$$\sigma_1 \alpha_l h = \sigma_1(h(\Psi_l^{-1}X)) = h^\sigma((\sigma\Psi_l^{-1})\Phi X) = h^\sigma(\Psi_{l+1}^{-1}X) = \alpha_{l+1}h^\sigma(X) = \alpha_{l+1}\sigma_0 h.$$

Therefore we have

$$\alpha_{l+1}^{-1}\sigma_1 = \sigma_0 \alpha_l^{-1}.$$

Since $\sigma_1 \mathfrak{p}_m \subset \mathfrak{p}_{m+1}$ by Lemma 4.4, we have an inclusion $\sigma_0 \alpha_l^{-1}\mathfrak{p}_{m,L_l} = \alpha_{l+1}^{-1}\sigma_1\mathfrak{p}_{m,L_l} \subset \alpha_{l+1}^{-1}\mathfrak{p}_{m+1,L_{l+1}}$. Replacing m by $m+l$, we obtain $\sigma_0\alpha_l^{-1}\mathfrak{p}_{m+l,L_l} \subset \alpha_{l+1}^{-1}\mathfrak{p}_{m+l+1,L_{l+1}}$. We consider the family of ideals $(\alpha_l^{-1}\mathfrak{p}_{m+l,L_l})_l$. Then for each l and m, we have $(\alpha_l^{-1}\mathfrak{p}_{m+l,L_l} \cap F[X, \Delta^{-1}])_{L_l} = \alpha_l^{-1}\mathfrak{p}_{m+l,L_l}$ by Lemmas 4.5, 4.6 and 4.7. Again, replacing m by $m-l$, we obtain an equality $(\alpha_l^{-1}\mathfrak{p}_{m,L_l} \cap F[X, \Delta^{-1}])_{L_l} = \alpha_l^{-1}\mathfrak{p}_{m,L_l}$.

We consider $L_l \otimes_E L_m$ as an L_l-algebra via $f \mapsto f \otimes 1$, and define an L_l-algebra homomorphism $\widetilde{\mu} : L_l[X, \Delta^{-1}] \to L_l \otimes_E L_m; X_{ij} \mapsto 1 \otimes \Psi_{ijm}$. Then $\mu_{lm} = \widetilde{\mu} \circ \alpha_l|_{F[X,\Delta^{-1}]}$ and the map

$$L_l[X, \Delta^{-1}] \twoheadrightarrow L_l[X, \Delta^{-1}]/\mathfrak{p}_{m,L_l} = L_l \otimes_E E[X, \Delta^{-1}]/\mathfrak{p}_m \xhookrightarrow{\mathrm{id}\otimes\mu_m} L_l \otimes_E L_m$$

coincides with $\widetilde{\mu}$. Therefore,

$$\mathfrak{q}_{m-l} = \mathfrak{q}_{l,m} = \ker \mu_{l,m} = \alpha_l|^{-1}_{F[X,\Delta^{-1}]}(\ker \widetilde{\mu}) = \alpha_l^{-1}\mathfrak{p}_{m,L_l} \cap F[X, \Delta^{-1}].$$

Thus we have $\mathfrak{q}_{m-l,L_l} = (\alpha_l^{-1}\mathfrak{p}_{m,L_l} \cap F[X, \Delta^{-1}])_{L_l} = \alpha_l^{-1}\mathfrak{p}_{m,L_l}$. □

Lemma 4.9. (1) *The ideal $\mathfrak{p} \subsetneq E[X, \Delta^{-1}]$ is maximal among the proper σ_1-invariant ideals.*

(2) *The family of ideals $(\mathfrak{p}_l)_l$ is maximal among the families of proper ideals $(\mathfrak{m}_l)_l$ of $E[X, \Delta^{-1}]$ which satisfies $\sigma_1\mathfrak{m}_l \subset \mathfrak{m}_{l+1}$ for all l.*

Proof. We prove only (2). Then (1) can be proved by the same argument. Let $(\mathfrak{m}_l)_l$ be a family of proper ideals of $E[X, \Delta^{-1}]$ such that $\mathfrak{p}_l \subset \mathfrak{m}_l$ and $\sigma_1\mathfrak{m}_l \subset \mathfrak{m}_{l+1}$ for all l. Let α_l be the homomorphism (4.1). We consider the family of ideals $(\alpha_l^{-1}\mathfrak{m}_{l,L_l})_l$. Since $\sigma_0\alpha_l^{-1}\mathfrak{m}_{l,L_l} = \alpha_{l+1}^{-1}\sigma_1\mathfrak{m}_{l,L_l} \subset \alpha_{l+1}^{-1}\mathfrak{m}_{l+1,L_{l+1}}$, we can apply Lemma 4.6 to $(\alpha_l^{-1}\mathfrak{m}_{l,L_l})_l$. Then $\alpha_l^{-1}\mathfrak{m}_{l,L_l} \cap F[X, \Delta^{-1}]$ is independent of l, and we take a maximal ideal $\mathfrak{a} \subset F[X, \Delta^{-1}]$ which contains this ideal. We also have $(\alpha_l^{-1}\mathfrak{m}_{l,L_l} \cap F[X, \Delta^{-1}])_{L_l} = \alpha_l^{-1}\mathfrak{m}_{l,L_l}$ by Lemmas 4.5, 4.6 and 4.7. Thus we obtain an inclusion $\alpha_l^{-1}\mathfrak{m}_{l,L_l} \subset \mathfrak{a}_{L_l}$. We put $M := F[X, \Delta^{-1}]/\mathfrak{a}$ and define a morphism

$$\pi_l : E[X, \Delta^{-1}] \hookrightarrow L_l[X, \Delta^{-1}] \xrightarrow[\alpha_l^{-1}]{} L_l[X, \Delta^{-1}] \xrightarrow[\rho_l]{} L_l[X, \Delta^{-1}]/\mathfrak{a}_{L_l} \xrightarrow[\beta_l]{} L_l \otimes_F M,$$

where ρ_l is the natural projection and $\beta_l : L_l[X, \Delta^{-1}]/\mathfrak{a}_{L_l} \cong L_l \otimes_F F[X, \Delta^{-1}]/\mathfrak{a} = L_l \otimes_F M$. Then we have $\mathfrak{m}_l \subset \ker \pi_l$.

We define a σ-action on $L \otimes_F M$ by $\sigma \otimes \mathrm{id}$. In $\mathrm{GL}_r(L_{l+1} \otimes_F M)$, we have

$$\begin{aligned}\sigma(\pi_l(X)) &= \sigma(\beta_l(\rho_l(\alpha_l^{-1}(X)))) = \beta_{l+1}(\rho_{l+1}(\sigma_0(\alpha_l^{-1}(X)))) = \beta_{l+1}(\rho_{l+1}(\alpha_{l+1}^{-1}(\sigma_1(X)))) \\ &= \pi_{l+1}(\sigma_1(X)) = \pi_{l+1}(\Phi X) = \Phi\pi_{l+1}(X).\end{aligned}$$

Set $\pi(X) := (\pi_l(X))_l \in \mathrm{GL}_r(L \otimes_F M)$. Then we have $\sigma(\pi(X)) = \Phi\pi(X)$. Since $(L \otimes_F M)^\sigma = M$, we obtain $\delta := \pi(X)^{-1}\Psi \in \mathrm{GL}_r((L \otimes_F M)^\sigma) = \mathrm{GL}_r(M)$. We define a δ-action on $(E \otimes_F M)[X, \Delta^{-1}]$ by $\delta \cdot h(X) := h(X\delta)$. We extend π_l to

$$\pi_l' : (E \otimes_F M)[X, \Delta^{-1}] = E[X, \Delta^{-1}] \otimes_F M \xrightarrow[\pi_l\otimes\mathrm{id}]{} L_l \otimes_F M.$$

Then we have $\mathfrak{p}_l \otimes_F M \subset \mathfrak{m}_l \otimes_F M \subset \ker \pi_l' = \delta \cdot \ker(\nu_l \otimes \mathrm{id}_M) = \delta \cdot (\mathfrak{p}_l \otimes_F M)$, where the first equality is proved as follows: For any $h(X) \in (E \otimes_F M)[X, \Delta^{-1}]$, $(\nu_l \otimes \mathrm{id}_M)(h(X\delta^{-1})) = h(\Psi_l\Psi^{-1}\pi(X)) = h(\pi_l(X)) = \pi_l'(h(X))$. Thus $h \in \ker \pi_l'$ is equivalent to $\delta^{-1} \cdot h \in \ker(\nu_l \otimes \mathrm{id}_M)$. Since $(E \otimes_F M)[X, \Delta^{-1}]$ is a noetherian ring, $\mathfrak{p}_l \otimes_F M \subset \delta \cdot (\mathfrak{p}_l \otimes_F M)$ implies $\mathfrak{p}_l \otimes_F M = \delta \cdot (\mathfrak{p}_l \otimes_F M)$. Therefore we have $\mathfrak{p}_l \otimes_F M = \mathfrak{m}_l \otimes_F M$. Since $(E \otimes_F M)[X, \Delta^{-1}]$ is faithfully flat over $E[X, \Delta^{-1}]$, we have $\mathfrak{p}_l = \mathfrak{m}_l$. □

Lemma 4.10. (1) *Let* $\mathfrak{b} \subset \Sigma[X, \Delta^{-1}]$ *be an ideal which is* σ_0*-invariant. Then we have* $\mathfrak{b} = (\mathfrak{b} \cap F[X, \Delta^{-1}])_\Sigma$.

(2) *Let* $\mathfrak{b}_l \subset \Sigma_l[X, \Delta^{-1}]$ *be ideals which satisfy* $\sigma_0 \mathfrak{b}_l \subset \mathfrak{b}_{l+1}$ *for all* l. *Then we have* $\mathfrak{b}_l = (\mathfrak{b}_l \cap F[X, \Delta^{-1}])_{\Sigma_l}$.

Proof. We prove only (2). Then (1) can be proved by the same argument. Set $\mathfrak{a} := \mathfrak{b}_l \cap F[X, \Delta^{-1}]$, which is independent of l by Lemma 4.7. Suppose that $\mathfrak{a}_{\Sigma_{l_0}} \subsetneq \mathfrak{b}_{l_0}$ for some l_0. Let $(g_i)_{i \in I}$ be an F-basis of $F[X, \Delta^{-1}]$ such that $I = I_1 \amalg I_\mathfrak{a}$ and $\mathfrak{a} = \oplus_{i \in I_\mathfrak{a}} F g_i$. Then we have

$$\Sigma_l[X, \Delta^{-1}] = (\oplus_{i \in I_1} \Sigma_l g_i) \oplus (\oplus_{i \in I_\mathfrak{a}} \Sigma_l g_i) = (\oplus_{i \in I_1} \Sigma_l g_i) \oplus \mathfrak{a}_{\Sigma_l}.$$

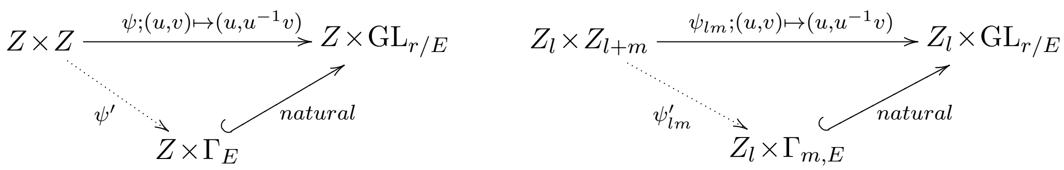
Since $\mathfrak{a}_{\Sigma_{l_0}} \subsetneq \mathfrak{b}_{l_0}$, we can take a minimal finite set $J \subset I_1$ so that $\mathfrak{b}_{l_0} \cap (\oplus_{i \in J} \Sigma_{l_0} g_i) \neq 0$. By the injectivity of σ_0 and the inclusion $\sigma_0(\mathfrak{b}_l \cap \oplus_{i \in J} \Sigma_l g_i) \subset \mathfrak{b}_{l+1} \cap \oplus_{i \in J} \Sigma_{l+1} g_i$, J has the same properties for all l. We fix $j \in J$ and consider the ideal of $\Sigma_l \cong E[X, \Delta^{-1}]/\mathfrak{p}_l$:

$$\mathfrak{m}_l := \{b \in \Sigma_l | \text{ there exists } \sum_{i \in J} b_i g_i \in \mathfrak{b}_l \cap (\oplus_{i \in J} \Sigma_l g_i) \text{ such that } b_j = b\}.$$

Then $\mathfrak{m}_l$ is a non-zero ideal by the minimality of J, and it is clear that $\sigma \mathfrak{m}_l \subset \mathfrak{m}_{l+1}$. By Lemma 4.4 we have $\sigma \nu_l = \nu_{l+1} \sigma_1 |_{E[X, \Delta^{-1}]}$. Hence we can apply Lemma 4.9 to the inverse image of $(\mathfrak{m}_l)_l$ in $E[X, \Delta^{-1}]$. Therefore we have $\mathfrak{m}_l = \Sigma_l$. Thus for each l, there exists an element $h_l = \sum_{i \in J} b_{li} g_i \in \mathfrak{b}_l \cap (\oplus_{i \in J} \Sigma_l g_i)$ such that $b_{lj} = 1$. Then we have $\mathfrak{b}_{l+1} \ni \sigma_0 h_l - h_{l+1} = \sum_{i \in J \setminus \{j\}} (\sigma b_{li} - b_{l+1,i}) g_i$. By the minimality of J, $\sigma b_{li} = b_{l+1,i}$ for all i and l. We put $b_i := (b_{li})_l \in \prod_l \Sigma_l$. Then we have $\sigma b_i = (\sigma b_{l-1,i})_l = (b_{li})_l = b_i$. Hence $b_i \in F$ and $b_{li} \in F$ via the l-th projection. Then $0 \neq h_l = \sum_{i \in J} b_{li} g_i \in \mathfrak{b}_l \cap F[X, \Delta^{-1}] = \mathfrak{a} = \oplus_{i \in I_\mathfrak{a}} F g_i$. This contradicts $J \cap I_\mathfrak{a} = \emptyset$. □

Proposition 4.11. (1) *Let* $\psi : Z \times_E Z \to Z \times_E \mathrm{GL}_{r/E}$ *be the morphism of affine* E*-schemes defined by* $(u, v) \mapsto (u, u^{-1} v)$ *for any* E*-algebra* S *and any* S*-valued point* $(u, v) \in Z(S) \times Z(S)$. *Then* ψ *factors through an isomorphism* $\psi' : Z \times_E Z \to Z \times_E \Gamma_E$ *of affine* E*-schemes.*

(2) *For any* l *and* m, *let* $\psi_{lm} : Z_l \times_E Z_{l+m} \to Z_l \times_E \mathrm{GL}_{r/E}$ *be the morphism of affine* E*-schemes defined by* $(u, v) \mapsto (u, u^{-1} v)$ *for any* E*-algebra* S *and any* S*-valued point* $(u, v) \in Z_l(S) \times Z_{l+m}(S)$. *Then* ψ *factors through an isomorphism* $\psi'_{lm} : Z_l \times_E Z_{l+m} \to Z_l \times_E \Gamma_{m,E}$ *of affine* E*-schemes.*



Proof. We prove only (2). Then (1) can be proved by the same argument. Let α_l and $\bar{\alpha}_{lm}$ be the homomorphisms (4.1) and (4.2). We restrict the domain of $\bar{\alpha}_{lm}$ to $\Sigma_l[X, \Delta^{-1}] \cong E[X, \Delta^{-1}]/\mathfrak{p}_l \otimes_E E[X, \Delta^{-1}]$ and the target of $\bar{\alpha}_{lm}$ to $\Sigma_l \otimes_E E[X, \Delta^{-1}]/\mathfrak{p}_m \cong E[X, \Delta^{-1}]/\mathfrak{p}_l \otimes_E E[X, \Delta^{-1}]/\mathfrak{p}_m$. Then ψ_{lm} corresponds to $\bar{\alpha}_{l,l+m}$ on the level of coordinate rings. Hence it is enough to show that $\alpha_l^{-1} \mathfrak{p}_{m+l, \Sigma_l} = \mathfrak{q}_{m, \Sigma_l}$.

Since we have an inclusion $\sigma_0 \alpha_l^{-1} \mathfrak{p}_{m+l, \Sigma_l} = \alpha_{l+1}^{-1} \sigma_1 \mathfrak{p}_{m+l, \Sigma_l} \subset \alpha_{l+1}^{-1} \mathfrak{p}_{m+l+1, \Sigma_{l+1}}$, the ideal $\mathfrak{a}_m := \alpha_l^{-1} \mathfrak{p}_{m+l, \Sigma_l} \cap F[X, \Delta^{-1}]$ is independent of l by Lemma 4.7, and we can apply Lemma 4.10 to $(\alpha_l^{-1} \mathfrak{p}_{m+l, \Sigma_l})_l$. Then we have $\alpha_l^{-1} \mathfrak{p}_{m+l, \Sigma_l} = (\alpha_l^{-1} \mathfrak{p}_{m+l, \Sigma_l} \cap F[X, \Delta^{-1}])_{\Sigma_l} =$

$\mathfrak{a}_{m,\Sigma_l}$. On the other hand we have $\alpha_l^{-1}\mathfrak{p}_{m+l,L_l} = \mathfrak{q}_{m,L_l}$ by Proposition 4.8. Therefore $\mathfrak{q}_{m,L_l} = \alpha_l^{-1}\mathfrak{p}_{m+l,L_l} = \alpha_l^{-1}((\mathfrak{p}_{m+l,\Sigma_l})_{L_l}) = (\alpha_l^{-1}\mathfrak{p}_{m+l,\Sigma_l})_{L_l} = (\mathfrak{a}_{m,\Sigma_l})_{L_l} = \mathfrak{a}_{m,L_l}$. Then we have $\mathfrak{q}_m = \mathfrak{a}_m$ since $L_l[X,\Delta^{-1}]$ is faithfully flat over $F[X,\Delta^{-1}]$. Thus we obtain $\alpha_l^{-1}\mathfrak{p}_{m+l,\Sigma_l} = \mathfrak{a}_{m,\Sigma_l} = \mathfrak{q}_{m,\Sigma_l}$. □

The next lemma is proved by an elementary argument. Thus we omit the proof. This lemma is applied to the S-valued points of the diagrams in Proposition 4.11 where S is an $\bar{E}$-algebra.

Lemma 4.12. (1) *Let G be a group, A and B be non-empty subsets of G such that the map*

$$\psi : A \times A \to A \times G;\ (u,v) \mapsto (u, u^{-1}v)$$

factors through a bijection $\psi' : A \times A \to A \times B$. Then B is a subgroup of G, A is stable under right-multiplication by elements of B and A becomes a B-torsor.

(2) *Let G be a group, A_l and B_m be non-empty subsets of G such that the map*

$$\psi_{lm} : A_l \times A_{l+m} \to A_l \times G;\ (u,v) \mapsto (u, u^{-1}v)$$

factors through a bijection $\psi'_{lm} : A_l \times A_{l+m} \to A_l \times B_m$ for each l and m. Then B_0 is a subgroup of G, A_l is stable under right-multiplication by elements of B_0 and A_l becomes a B_0-torsor for each l. Moreover, for any $u \in A_l$ and $v \in A_{l+m}$ there exists an element $y \in B_m$ such that $v = uy$. The multiplication in G induces maps $A_l \times B_m \to A_{l+m}$ and $B_m \times B_{m'} \to B_{m+m'}$, and the inversion in G induces a map $B_m \to B_{-m}$.

By Proposition 4.11 and Lemma 4.12, we have surjective maps

$$Z_l(S) \times Z_{l+m}(S) \to \Gamma_m(S);\ (u,v) \mapsto u^{-1}v, \tag{4.3}$$

$$Z_l(S) \times \Gamma_m(S) \to Z_{l+m}(S);\ (x,y) \mapsto xy \tag{4.4}$$

for any $\bar{E}$-algebra S.

Theorem 4.13. (1) *The F-scheme Γ is a closed F-subgroup scheme of $\mathrm{GL}_{r/F}$, the E-scheme Z is stable under right multiplication by Γ_E and is a Γ_E-torsor.*

(2) *The F-scheme Γ_0 is a closed F-subgroup schemes of $\mathrm{GL}_{r/F}$, the E-scheme Z_l is stable under right multiplication by $\Gamma_{0,E}$ and is a $\Gamma_{0,E}$-torsor for each l.*

(3) *The F-scheme Γ_m is stable under right and left multiplications by Γ_0 and is a Γ_0-torsor for each m.*

Proof. We prove only (2). Then (1) and (3) can be proved by the same argument. By Proposition 4.11, we have a bijection $Z_l(S)\times Z_{l+m}(S) \to Z_l(S)\times\Gamma_m(S); (u,v) \mapsto (u,u^{-1}v)$ for any $\bar{E}$-algebra S. Since $Z_l(S)$ is non-empty, Lemma 4.12 implies that $\Gamma_{0,\bar{E}}$ is a closed subgroup scheme of $\mathrm{GL}_{r/\bar{E}}$ and $Z_{l,\bar{E}}$ is a $\Gamma_{0,\bar{E}}$-torsor. Therefore Γ_0 is a closed subgroup scheme of $\mathrm{GL}_{r/F}$ by the faithfully flatness of the inclusion $F \to \bar{E}$. Similarly, Z_l is a $\Gamma_{0,E}$-torsor by the faithfully flatness of the inclusion $E \to \bar{E}$. □

Theorem 4.14. (*a*) *The E-schemes Z and Z_l are smooth.*

(*b*) *The F-schemes Γ and Γ_m are smooth.*

(*c*) *If E is algebraically closed in the fraction field Λ_{l_0} of Σ_{l_0} for some l_0, then Z_l and Γ_m are absolutely irreducible.*

(*d*) $\dim\Gamma = \dim\Gamma_m = \mathrm{tr.deg}_E\,\Lambda_l$.

Proof. (a), (b) Since L_l/E is a separable extension, Λ/E is also a separable extension, where $\Lambda = \mathrm{Frac}(\Sigma)$ the total ring of fractions of Σ. Thus for any field extension Ω/E, $\Lambda \otimes_E \Omega$ is reduced. Therefore $\Sigma \otimes_E \Omega$ is reduced and $Z = \mathrm{Spec}\,\Sigma$ is absolutely reduced. Since $\Gamma_{\bar{E}} \cong Z_{\bar{E}}$, Γ is absolutely reduced. Since Γ is an algebraic group, the property that Γ is absolutely reduced implies that Γ is smooth. Again since $\Gamma_{\bar{E}} \cong Z_{\bar{E}}$, we have that Z is smooth. The statements of Z_l and Γ_m are proved similarly.

(c) For any field extension Ω/E, $\Lambda_{l_0} \otimes_E \Omega$ is an integral domain by the assumption. Therefore $\Sigma_{l_0} \otimes_E \Omega$ is an integral domain and Z_{l_0} is absolutely integral. Since $Z_{l,\bar{E}} \cong \Gamma_{m,\bar{E}}$ for all l and m, Z_l and Γ_m are all absolutely integral.

(d) We have an equality $\dim \Gamma = \dim \Gamma_m = \dim \Gamma_0 = \dim Z_l = \mathrm{tr.deg}_E \Lambda_l$. □

Corollary 4.15. (1) *There exists a divisor d' of d such that if $l \equiv l' \pmod{d'}$ then $Z_l = Z_{l'}$ and if $l \not\equiv l' \pmod{d'}$ then $Z_l \cap Z_{l'} = \emptyset$.*

(2) *If $m \equiv m' \pmod{d'}$ then $\Gamma_m = \Gamma_{m'}$ and if $m \not\equiv m' \pmod{d'}$ then $\Gamma_m \cap \Gamma_{m'} = \emptyset$.*

Therefore we can write $Z = \amalg_{l \in \mathbb{Z}/d'} Z_l$, $\Sigma = \prod_{l \in \mathbb{Z}/d'} \Sigma_l$, $\Lambda = \prod_{l \in \mathbb{Z}/d'} \Lambda_l$ and $\Gamma = \amalg_{m \in \mathbb{Z}/d'} \Gamma_m$.

Proof. (1) Since Z_l is a $\Gamma_{0,E}$-torsor and absolutely reduced for all l, it is clear that $Z_l = Z_{l'}$ or $Z_l \cap Z_{l'} = \emptyset$. We have the surjective map (4.4) : $Z_l(\bar{E}) \times \Gamma_1(\bar{E}) \to Z_{l+1}(\bar{E})$. Therefore if $Z_l = Z_{l'}$, then $Z_{l+1}(\bar{E}) = Z_{l'+1}(\bar{E})$. Hence if we take d' to be the minimum positive integer such that $Z_0 = Z_{d'}$, then d' satisfies the desired properties.

(2) By the same argument of the proof of (1), there exists a divisor d'' of d which is the period of $(\Gamma_m)_m$. Then by the map (4.3), we have

$$\Gamma_m(\bar{E}) = Z_l(\bar{E})^{-1} Z_{l+m}(\bar{E}) = Z_l(\bar{E})^{-1} Z_{l+m+d'}(\bar{E}) = \Gamma_{m+d'}(\bar{E}).$$

This means that $d''|d'$. By the map (4.4), we have

$$Z_l(\bar{E}) = Z_l(\bar{E})\Gamma_0(\bar{E}) = Z_l(\bar{E})\Gamma_{d''}(\bar{E}) = Z_{l+d''}(\bar{E}).$$

This means that $d'|d''$. □

4.2 Γ-action

For any F-algebras R and S, we set $S^{(R)} := R \otimes_F S$. In particular, if $R = F'$ is a field, we set $S' := S^{(F')}$. If σ acts on S, we define the σ-action on $S^{(R)}$ by $\mathrm{id} \otimes \sigma$. Note that, if $S^\sigma = F$, then we have $(S^{(R)})^\sigma = R$. Let $\mathrm{Aut}_\sigma(\Sigma^{(R)}/E^{(R)})$ denote the group of automorphisms of $\Sigma^{(R)}$ over $E^{(R)}$ that commute with σ. Similarly we define $\mathrm{Aut}_\sigma(\Lambda^{(R)}/E^{(R)})$. For any $\gamma \in \Gamma(R)$, we obtain an automorphism $Z_{E^{(R)}} \to Z_{E^{(R)}}; x \mapsto x\gamma$. On the level of coordinate rings, this corresponds to an automorphism $\Sigma^{(R)} \to \Sigma^{(R)}; h(\Psi) \mapsto \gamma.h(\Psi) := h(\Psi\gamma)$. Note that $\Sigma^{(R)} = E^{(R)} \otimes_E \Sigma = E^{(R)}[\Psi, \Delta(\Psi)^{-1}] \cong E^{(R)}[X, \Delta^{-1}]/\mathfrak{p}_{E^{(R)}}$. Thus we have a group homomorphism $\kappa_R : \Gamma(R) \to \mathrm{Aut}(\Sigma^{(R)}/E^{(R)})$.

Lemma 4.16. (1) *For any F-algebra R, the map κ_R induces an isomorphism $\Gamma(R) \xrightarrow{\sim} \mathrm{Aut}_\sigma(\Sigma^{(R)}/E^{(R)})$. Its inverse is the map $\alpha \mapsto \Psi^{-1}(\alpha\Psi_{ij})_{ij}$.*

(2) $\mathrm{Aut}_\sigma(\Sigma/E) \cong \mathrm{Aut}_\sigma(\Lambda/E)$.

(3) *If Λ_l/F is a regular extension (i.e. separable extension and F is algebraically closed in Λ_l) for all l and F'/F is an algebraic extension of fields, then we have $\mathrm{Aut}_\sigma(\Sigma'/E') \cong \mathrm{Aut}_\sigma(\Lambda'/E')$.*

Proof. (1) For any $\gamma \in \Gamma(R)$ and $h(\Psi) \in \Sigma^{(R)}$, we have $\sigma(\gamma.h(\Psi)) = \sigma(h(\Psi\gamma)) = h^{\sigma}(\sigma(\Psi\gamma)) = h^{\sigma}(\Phi\Psi\gamma) = \gamma.(h^{\sigma}(\Phi\Psi)) = \gamma.(h^{\sigma}(\sigma\Psi)) = \gamma.(\sigma(h(\Psi)))$. Hence $\kappa_R(\gamma)$ commutes with σ. Suppose that $\kappa_R(\gamma)$ is the identity. Then $h(\Psi\gamma) = h(\Psi)$ for any $h(\Psi) \in \Sigma^{(R)}$. In particular if we take $h(\Psi) = \Psi_{ij}$ for each i and j, then we obtain $\Psi\gamma = \Psi$ in $\mathrm{GL}_r(\Sigma^{(R)})$. Therefore $\gamma = 1$ and this means that κ_R is injective. Conversely, let $\alpha \in \mathrm{Aut}_{\sigma}(\Sigma^{(R)}/E^{(R)})$ be any element. Then α corresponds to an automorphism $\bar{\alpha} : Z_{E^{(R)}} \to Z_{E^{(R)}}$, and $\bar{\alpha}$ maps the $\Sigma^{(R)}$-valued point Ψ to $(\alpha\Psi_{ij})_{ij}$. By Theorem 4.13, there exists an element $\gamma \in \Gamma(\Sigma^{(R)})$ such that $\Psi\gamma = (\alpha\Psi_{ij})_{ij}$. Then for any $h(\Psi) \in \Sigma^{(R)}$, we have $\alpha(h(\Psi)) = h((\alpha\Psi_{ij})_{ij}) = h(\Psi\gamma)$. Thus we obtain $\sigma(\gamma.h(\Psi)) = \sigma(\alpha(h(\Psi))) = \alpha(\sigma(h(\Psi))) = \alpha(h^{\sigma}(\Phi\Psi)) = \gamma.h^{\sigma}(\Phi\Psi) = h^{\sigma}(\Phi\Psi\gamma)$. On the other hand, $\sigma(\gamma.h(\Psi)) = \sigma(h(\Psi\gamma)) = h^{\sigma}((\sigma\Psi)(\sigma\gamma)) = h^{\sigma}(\Phi\Psi(\sigma\gamma))$. If we take $h(\Psi) = \Psi_{ij}$ for each i and j, we obtain $\Phi\Psi(\sigma\gamma) = \Phi\Psi\gamma$. Hence $\sigma\gamma = \gamma$. Therefore we have $\gamma \in \Gamma(R)$ and $\kappa_R(\gamma) = \alpha$.

(2) Since $\Lambda = \mathrm{Frac}(\Sigma)$, any automorphism of Σ extends uniquely to an automorphism of Λ. Conversely if $\alpha \in \mathrm{Aut}_{\sigma}(\Lambda/E)$, then $\sigma(\alpha(\Psi)) = \alpha(\sigma\Psi) = \alpha(\Phi\Psi) = \Phi \cdot \alpha(\Psi)$ in $\mathrm{GL}_r(L)$. Thus we have $\alpha(\Psi) \in \Psi \cdot \mathrm{GL}_r(F)$. This implies that $\alpha(\Sigma) \subset \Sigma$. Similarly, we have $\alpha^{-1}(\Sigma) \subset \Sigma$. Therefore $\alpha(\Sigma) = \Sigma$.

(3) Since Λ_l/F is a regular extension, so is E/F. Since F'/F is an algebraic extension, E' and Λ'_l are fields and $\Lambda'_l = \mathrm{Frac}(\Sigma'_l)$. Therefore $\Lambda' = \prod_{l \in \mathbb{Z}/d'} \Lambda'_l$ is a finite product of fields and $\Lambda' = \mathrm{Frac}(\Sigma')$. Then, the proof is the same as (2). □

We prepare some lemmas about Zariski density.

Lemma 4.17. *Let Ω/k be a field extension such that Ω is an algebraically closed field, X an algebraic variety over k and Y a closed subvariety of X_{Ω}. If $X(k) \cap Y(\Omega)$ is Zariski dense in Y, then Y is defined over k, i.e. there exists some algebraic variety Y_0 over k such that $Y = Y_{0,\Omega}$.*

Proof. We may assume that X is affine. Let $k[X]$ be the coordinate ring of X and $\Omega[X]$ the coordinate ring of X_{Ω}. Then we have $\Omega[X] = \Omega \otimes_k k[X]$. Let $\mathfrak{a} \subset \Omega[X]$ be the defining ideal of Y, $\mathfrak{a}_k := \mathfrak{a} \cap k[X]$ and $\mathfrak{a}_{k,\Omega} := \mathfrak{a}_k \cdot \Omega[X]$. We need to show that $\mathfrak{a}_{k,\Omega} = \mathfrak{a}$. Thus we assume that $\mathfrak{a}_{k,\Omega} \subsetneq \mathfrak{a}$. Let $(g_i)_{i \in I'}$ be a k-basis of $k[X]$ such that $(g_i)_{i \in I}$ is a k-basis of $\mathfrak{a}_k$ for some $I \subset I'$. We also take $(c_j)_{j \in J}$ to be a k-basis of Ω. Since $\mathfrak{a}_{k,\Omega} \subsetneq \mathfrak{a}$, there exists a non-zero element $f = \sum_{i \in I' \setminus I} a_i g_i \in \mathfrak{a}$ where $a_i \in \Omega$. Write $a_i = \sum_j \alpha_{ij} c_j$ ($\alpha_{ij} \in k$). Then we can write $f = \sum_j c_j \sum_{i \in I' \setminus I} \alpha_{ij} g_i$. For any $x \in X(k) \cap Y(\Omega)$, we have $\sum_j c_j \sum_{i \in I' \setminus I} \alpha_{ij} g_i(x) = f(x) = 0$. Since $(c_j)_j$ is linearly independent over k, we have $\sum_{i \in I' \setminus I} \alpha_{ij} g_i(x) = 0$ for all j. By the density assumption, we obtain $\sum_{i \in I' \setminus I} \alpha_{ij} g_i(x) = 0$ for all $x \in Y(\Omega)$. Therefore $\sum_{i \in I' \setminus I} \alpha_{ij} g_i \in \mathfrak{a} \cap k[X] = \mathfrak{a}_k = \oplus_{i \in I} k g_i$ for all j. Since $(g_i)_{i \in I'}$ is linearly independent over k, we have $\alpha_{ij} = 0$. Thus $f = 0$, which is a contradiction. □

Corollary 4.18. *Let Ω/k be a field extension such that Ω is an algebraically closed field and X an algebraic variety over k. If $X(k)$ is Zariski dense in X, then $X(k)$ is Zariski dense in X_{Ω}.*

Proof. We take Y to be the Zariski closure of $X(k)$ in X_{Ω}. Then there exists some algebraic variety Y_0 over k such that $Y = Y_{0,\Omega}$ by Lemma 4.17. It is clear that $X(k) \subset Y_0(k)$. Hence we have $Y_0 = X$ since $X(k)$ is Zariski dense in X. Therefore we have $Y = X_{\Omega}$. □

Lemma 4.19. *Let Ω/k be a field extension, X_1 an algebraic variety over Ω and X_2 an algebraic varieties over k. If $X_2(k)$ is Zariski dense in X_2, then $X_1(\bar{\Omega}) \times X_2(k)$ is Zariski dense in $X_1 \times_\Omega X_{2,\Omega}$, where $\bar{\Omega}$ is an algebraic closure of Ω.*

Proof. Let V be the Zariski closure of $X_1(\bar{\Omega}) \times X_2(k)$ in $X_1 \times_\Omega X_{2,\Omega}$. We assume that $V \subsetneq X_1 \times_\Omega X_{2,\Omega}$. Then we have an element $(x, y) \in (X_1(\bar{\Omega}) \times X_2(\bar{\Omega})) \setminus V(\bar{\Omega})$. Therefore we have $(\{x\} \times X_{2,\bar{\Omega}}) \cap V_{\bar{\Omega}} \subsetneq \{x\} \times X_{2,\bar{\Omega}}$ and $\{x\} \times X_2(k) \subset (\{x\} \times X_2(\bar{\Omega})) \cap V(\bar{\Omega})$. On the other hand, since $X_2(k)$ is Zariski dense in X_2, $X_2(k)$ is Zariski dense in $X_{2,\bar{\Omega}}$ by Corollary 4.18. Then $\{x\} \times X_2(k)$ is Zariski dense in $\{x\} \times_{\bar{\Omega}} X_{2,\bar{\Omega}}$. This is a contradiction. □

Theorem 4.20. *Let F'/F be an algebraic extension of fields such that $\Gamma(F')$ is Zariski dense in $\Gamma_{F'}$. Assume that $F' = F$ or Λ_l/F is a regular extension for all l. Then we have $(\Lambda')^{\Gamma(F')} = E'$ and $\Lambda \cap (\Lambda')^{\Gamma(F')} = E$.*

Proof. The second part follows from the first part and the assumptions. Thus we prove the first part. In the proof of this theorem, we regard l as an element of the index set $\mathbb{Z}/d'$. We take any element $f = (f_l)_l \in (\Lambda')^{\Gamma(F')} \subset \prod_{l \in \mathbb{Z}/d'} \Lambda'_l$, and consider f_l as a rational function of $Z_{l,E'}$ to $\mathbb{A}^1_{E'}$. Then, for some non-empty open affine set $U_l \subset Z_{l,E'}$, f_l can be regarded as a morphism $f_l : U_l \to \mathbb{A}^1_{E'}$. By Proposition 4.11, we have an isomorphism $Z_{E'} \times_{E'} \Gamma_{E'} \to Z_{E'} \times_{E'} Z_{E'}; (x, y) \mapsto (x, xy)$. We set $U \subset Z_{E'} \times_{E'} \Gamma_{E'}$ to be the open subset corresponding to $\amalg_l U_l \times_{E'} \amalg_l U_l$ via this isomorphism, and consider the two maps

$$g_i : U \longrightarrow \amalg U_l \times_{E'} \amalg U_l \xrightarrow{\pi_i} \amalg U_l \xrightarrow{f=(f_l)} \mathbb{A}^1_{E'}$$

where $i = 1, 2$ and π_i is the i-th projection. Let S be an algebraic closure of E'. Then for any $(x, y) \in (Z(S) \times \Gamma(F')) \cap U(S)$, we have $g_1(x, y) = f(\pi_1(x, xy)) = f(x)$ and $g_2(x, y) = f(\pi_2(x, xy)) = f(xy) = f(x)$ since f is fixed by $\Gamma(F')$. Since $\Gamma(F')$ is Zariski dense in $\Gamma_{F'}$, $Z(S) \times \Gamma(F')$ is Zariski dense in $Z_{E'} \times_{E'} \Gamma_{E'}$ by Lemma 4.19. Then $(Z(S) \times \Gamma(F')) \cap U(S)$ is Zariski dense in U. Thus we have $g_1 = g_2$, and this means $f\pi_1 = f\pi_2$. By considering on the level of coordinate rings, it is clear that $f \in E'$ since E' is a field. □

Corollary 4.21. *If F is a local field, and each connected component of Γ has an F-valued point, then $\Lambda^{\Gamma(F)} = E$.*

Proof. Take any connected component Γ' of Γ. Then there exists an F-valued point $x \in \Gamma'(F)$ by the assumption, and Γ' is smooth by Lemma 4.14. By the implicit function theorem, there exists an open neighborhood of x in $\Gamma'(F)$ which is isomorphic to some open subset of $F^{\dim \Gamma}$. Since Γ' is irreducible, this implies that $\Gamma'(F)$ is Zariski dense in Γ'. Hence we conclude that $\Gamma(F)$ is Zariski dense in Γ. Then this corollary follows from Theorem 4.20. □

5 The group Γ and φ-modules

5.1 General case

In this subsection, we use the notations defined in Section 3, and fix a σ-admissible triple (F, E, L). Let $M \in \Phi\mathrm{M}^L_E$ be an L-trivial φ-module over E of rank r, $\mathcal{T}_M$ the Tannakian subcategory of $\Phi\mathrm{M}^L_E$ generated by M, $V_M : \mathcal{T}_M \to \mathbf{Vec}(F)$ the fiber functor of $\mathcal{T}_M$ and Γ_M the Tannakian Galois group of $(\mathcal{T}_M, V_M)$. We fix $\mathbf{m} \in \mathrm{Mat}_{r\times 1}(M)$ an E-basis of M. Then there exist matrices $\Phi \in \mathrm{GL}_r(E)$ and $\Psi \in \mathrm{GL}_r(L)$ such that $\varphi\mathbf{m} = \Phi\mathbf{m}$ and $\sigma\Psi = \Phi\Psi$. We define $\Gamma, \Sigma, \dots$ as in Section 4 for Φ and Ψ. In this subsection, we show that there exists an equivalence of categories $\mathcal{T}_M \xrightarrow{\sim} \mathbf{Rep}(\Gamma, F)$ under some assumptions. Note that Σ and Σ_l are independent of the choice of $\mathbf{m}$ and Ψ by Proposition 3.13. If $N \in \mathcal{T}_M$ and s is the rank of N over E, we use the notation $\mathbf{n} \in \mathrm{Mat}_{s\times 1}(N)$ and $\Psi_N \in \mathrm{GL}_s(L)$ for an E-basis of N and a fundamental matrix respectively. For any F-algebras S and R, we set $S^{(R)} := R \otimes_F S$.

Proposition 5.1. *For any $N \in \mathcal{T}_M$, we have $\Psi_N \in \mathrm{GL}_s(\Sigma)$.*

Proof. Let N and N' be objects in $\mathcal{T}_M$. Set $s := \dim_E N$ and $s' := \dim_E$ and assume that $\Psi_N \in \mathrm{GL}_s(\Sigma)$ and $\Psi_{N'} \in \mathrm{GL}_{s'}(\Sigma)$. Since we can take $\Psi_{N\oplus N'} = \Psi_N \oplus \Psi_{N'}$, $\Psi_{N\otimes N'} = \Psi_N \otimes \Psi_{N'}$ and $\Psi_{N^\vee} = (\Psi_N^{-1})^{\mathrm{tr}}$, we have that $\Psi_{N\oplus N'}$, $\Psi_{N\otimes N'}$ and $\Psi_{N^\vee}$ are invertible matrices with coefficients in Σ. We have to show that if $0 \to N' \to N \to N'' \to 0$ is an exact sequence in $\mathcal{T}_M$ and $\Psi_N \in \mathrm{GL}_s(\Sigma)$, then $\Psi_{N'} \in \mathrm{GL}_{s'}(\Sigma)$ and $\Psi_{N''} \in \mathrm{GL}_{s''}(\Sigma)$. Let $\mathbf{n}$, $\mathbf{n}'$ and $\mathbf{n}''$ be E-bases of N, N' and N'' such that

$$\mathbf{n} = \begin{bmatrix} \mathbf{n}' \\ \tilde{\mathbf{n}}'' \end{bmatrix}$$

where $\tilde{\mathbf{n}}''$ is a lift of $\mathbf{n}''$. Since V_M is exact, we have an exact sequence

$$0 \to V_M(N') \to V_M(N) \to V_M(N'') \to 0.$$

Let $\mathbf{x}$, $\mathbf{x}'$ and $\mathbf{x}''$ be F-bases of $V_M(N)$, $V_M(N')$ and $V_M(N'')$ such that

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}' \\ \tilde{\mathbf{x}}'' \end{bmatrix}$$

where $\tilde{\mathbf{x}}''$ is a lift of $\mathbf{x}''$. By Proposition 3.14, there exist matrices $A \in \mathrm{GL}_s(F)$, $A' \in \mathrm{GL}_{s'}(F)$ and $A'' \in \mathrm{GL}_{s''}(F)$ such that $\Psi_N^{-1}\mathbf{n} = A\mathbf{x}$, $\Psi_{N'}^{-1}\mathbf{n}' = A'\mathbf{x}'$ and $\Psi_{N''}^{-1}\mathbf{n}'' = A''\mathbf{x}''$. Consider the exact sequence

$$0 \to L \otimes_E N' \to L \otimes_E N \to L \otimes_E N'' \to 0.$$

Since both $\Psi_{N''}A''\tilde{\mathbf{x}}''$ and $\tilde{\mathbf{n}}''$ are mapped to $\mathbf{n}''$ and $\mathbf{x}'$ is an L-basis of $L \otimes_E N'$, there exists a matrix $B \in \mathrm{Mat}_{s''\times s'}(L)$ such that

$$\tilde{\mathbf{n}}'' = B\mathbf{x}' + \Psi_{N''}A''\tilde{\mathbf{x}}''.$$

Therefore we have

$$\Psi_N A\mathbf{x} = \mathbf{n} = \begin{bmatrix} \mathbf{n}' \\ \tilde{\mathbf{n}}'' \end{bmatrix} = \begin{bmatrix} \Psi_{N'}A' & 0 \\ B & \Psi_{N''}A'' \end{bmatrix} \mathbf{x}.$$

Since $\Psi_N \in \mathrm{GL}_s(\Sigma)$, we conclude that $\Psi_{N'} \in \mathrm{GL}_{s'}(\Sigma)$ and $\Psi_{N''} \in \mathrm{GL}_{s''}(\Sigma)$. □

Lemma 5.2. *For any $N \in \mathcal{T}_M$ and F-algebra R, there exists a natural isomorphism*

$$\Sigma^{(R)} \otimes_F V(N) \to \Sigma^{(R)} \otimes_E N.$$

Similarly, there exists a natural isomorphism

$$\Sigma_l^{(R)} \otimes_F V(N) \to \Sigma_l^{(R)} \otimes_E N$$

for all l.

Proof. The inclusion $V(N) \subset \Sigma \otimes_E N$ and the product map $\Sigma \otimes_F \Sigma \to \Sigma$ induce a natural map

$$\kappa : \Sigma^{(R)} \otimes_F V(N) \hookrightarrow \Sigma^{(R)} \otimes_F \Sigma \otimes_E N \to \Sigma^{(R)} \otimes_E N.$$

Since $1 \otimes \Psi_N^{-1}\mathbf{n}$ is a $\Sigma^{(R)}$-basis of $\Sigma^{(R)} \otimes_F V(N)$, we can write κ explicitly as follows:

$$\kappa(\mathbf{f} \cdot (1 \otimes \Psi_N^{-1}\mathbf{n})) = (\mathbf{f}\Psi_N^{-1}) \cdot (1 \otimes \mathbf{n})$$

for all $\mathbf{f} \in \mathrm{Mat}_{1\times s}(\Sigma^{(R)})$. Hence it is clear that κ is an isomorphism. The Σ_l version is proved by the same argument. □

Theorem 5.3. *For any $N \in \mathcal{T}_M$, there exists a natural representation*

$$\rho_N : \Gamma \to \mathrm{GL}(V(N))$$

over F that is functorial in N.

Proof. For any F-algebra R and $\gamma \in \Gamma(R) \subset \mathrm{GL}_r(R)$, we define

$$\rho_N^{(R)}(\gamma) : R \otimes_F V(N) \hookrightarrow \Sigma^{(R)} \otimes_F V(N) \to \Sigma^{(R)} \otimes_E N \to \Sigma^{(R)} \otimes_E N,$$

where the second map is the isomorphism defined in Lemma 5.2 and the third map is defined by $h(\Psi) \otimes x \mapsto h(\Psi\gamma) \otimes x$. Clearly $\rho_N^{(R)}$ is functorial in N. If $\mathrm{im}(\rho_M^{(R)}(\gamma)) = R \otimes_F V(M)$ then $\mathrm{im}(\rho_N^{(R)}(\gamma)) = R \otimes_F V(N)$ for all $N \in \mathcal{T}_M$. Thus we may assume that $N = M$. We can write $\rho_M^{(R)}(\gamma)$ explicitly:

$$\rho_M^{(R)}(\gamma)(\mathbf{f} \cdot (1 \otimes \Psi^{-1}\mathbf{m})) = \mathbf{f}\gamma^{-1}(1 \otimes \Psi^{-1}\mathbf{m}),$$

for each $\mathbf{f} \in \mathrm{Mat}_{1\times r}(R)$. Therefore we have $\mathrm{im}(\rho_M^{(R)}(\gamma)) = R \otimes_F V(M)$. □

From the above description of $\rho_M^{(R)}$, we have the following corollary:

Corollary 5.4. *The representation $\rho_M : \Gamma \to \mathrm{GL}(V(M))$ is faithful.*

From Theorem 5.3, we have a functor $\xi_M : \mathcal{T}_M \to \mathbf{Rep}(\Gamma, F)$, and it is clear by the construction that ξ_M is a tensor functor. Let $\eta_M : \mathbf{Rep}(\Gamma_M, F)) \to \mathcal{T}_M$ be the equivalence of categories defined by the Tannakian duality and $\alpha : \mathbf{Rep}(\Gamma, F) \to \mathbf{Vec}(F)$ the forgetful functor. Since $V_M = \alpha \circ \xi_M$, there exists a unique homomorphism $\pi_M : \Gamma \to \Gamma_M$ over F such that the natural functor $\tau_M : \mathbf{Rep}(\Gamma_M, F) \to \mathbf{Rep}(\Gamma, F)$ induced by π_M satisfies $\xi_M \circ \eta_M = \tau_M$.

$$\begin{array}{ccccc} \mathbf{Rep}(\Gamma_M, F) & \xrightarrow{\eta_M} & \mathcal{T}_M & \xrightarrow{\xi_M} & \mathbf{Rep}(\Gamma, F) \\ & & & \searrow^{V_M} & \downarrow{\scriptstyle \alpha} \\ & & & & \mathbf{Vec}(F) \end{array}$$

Proposition 5.5. *For any representation $W \in \mathbf{Rep}(\Gamma, F)$, there exists an object $N \in \mathcal{T}_M$ such that W is isomorphic to a subquotient of $\xi_M(N)$.*

Proof. By Corollary 5.4, the Γ-representation $\xi_M(M) = \rho_M$ is faithful. Therefore, W is isomorphic to a subquotient of representation of the form

$$\oplus_{i=1}^n (\xi_M(M))^{\otimes a_i} \otimes (\xi_M(M)^\vee)^{\otimes b_i},$$

where $a_i, b_i \in \mathbb{N}$. However we have $\oplus_{i=1}^n (\xi_M(M))^{\otimes a_i} \otimes (\xi_M(M)^\vee)^{\otimes b_i} = \xi_M(\oplus_{i=1}^n M^{\otimes a_i} \otimes (M^\vee)^{\otimes b_i})$. □

Proposition 5.5 is equivalent to the next theorem ([6], Proposition 2.21).

Theorem 5.6. *The morphism of affine F-schemes $\pi_M : \Gamma \to \Gamma_M$ is a closed immersion.*

From now on, we assume that $\Gamma(F)$ is Zariski dense in Γ or Λ_l/F is a regular extension for each l. In the former case we put $F' = F$, and in the latter case we put $F' = \bar{F}$. For any F-algebra S, we set $S' := F' \otimes_F S$. Then in any case, E' and Λ'_l are fields, $\Lambda'_l = \mathrm{Frac}(\Sigma'_l)$ and $\Lambda \cap (\Lambda')^{\Gamma(F')} = E$ by Theorem 4.20.

Proposition 5.7. *Assume that $\Gamma(F)$ is Zariski dense in Γ or Λ_l/F is a regular extension for each l. Then the functor $\xi_M : \mathcal{T}_M \to \mathbf{Rep}(\Gamma, F)$ is fully faithful.*

Proof. For any objects $N, N' \in \mathcal{T}_M$, there exist natural isomorphisms $\mathrm{Hom}_{\mathcal{T}_M}(N', N) \cong \mathrm{Hom}_{\mathcal{T}_M}(\mathbf{1}, \mathrm{Hom}(N', N))$ and $\mathrm{Hom}_\Gamma(V(N'), V(N)) \cong \mathrm{Hom}_\Gamma(V(\mathbf{1}), V(\mathrm{Hom}(N', N)))$. Thus it is enough to show that, for any $N \in \mathcal{T}_M$, $\mathrm{Hom}_{\mathcal{T}_M}(\mathbf{1}, N) \to \mathrm{Hom}_\Gamma(V(\mathbf{1}), V(N))$ is an isomorphism. It is injective since $\mathrm{Hom}_{\mathcal{T}_M}(\mathbf{1}, N) = N^\varphi = N \cap V(N) \hookrightarrow \mathrm{Hom}_\Gamma(V(\mathbf{1}), V(N))$. For any $\phi \in \mathrm{Hom}_\Gamma(V(\mathbf{1}), V(N))$, there exists $\mathbf{h} = \mathbf{h}(\Psi) \in \mathrm{Mat}_{1\times s}(\Sigma)$ so that $\phi(1) = \mathbf{hn}$ by Lemma 5.2. Then for any $\gamma \in \Gamma(F')$, we have $\mathbf{h}(\Psi)\mathbf{n} = \phi(1) = \gamma.\phi(1) = \mathbf{h}(\Psi\gamma)\mathbf{n}$. Hence $\mathbf{h}(\Psi) = \mathbf{h}(\Psi\gamma) = \gamma.\mathbf{h}$. By Theorem 4.20, we have $\mathbf{h} \in \mathrm{Mat}_{1\times s}(E)$, and this implies $\phi(1) = \mathbf{hn} \in N \cap V(N)$. □

We prepare a lemma from linear algebra.

Lemma 5.8. *Let $E \subset \Lambda$ be general rings where E is a field and $\Lambda = \prod_{l\in\mathbb{Z}/d'} \Lambda_l$ is a finite product of fields. Assume that $\#E > d'$. Let $1 \le m \le s$ and $D \in \mathrm{Mat}_{s\times m}(\Lambda)$. If there exists $D_0 \in \mathrm{GL}_s(\Lambda)$ such that $D_0 = [*, D]$, then there exist $A \in \mathrm{GL}_m(\Lambda)$ and $B \in \mathrm{GL}_s(E)$ such that*

$$BDA = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ * & * & * \end{bmatrix} \in \mathrm{Mat}_{s\times m}(\Lambda).$$

Proof. For each $l \in \mathbb{Z}/d'$ and $1 \le j \le m$, let $e_{l,j} \in \mathrm{Mat}_{1\times m}(\Lambda_l)$ be a row vector such that the j-th component is one and the other components are zero. Write $D = (D_l)_l$ where $D_l \in \mathrm{Mat}_{s\times m}(\Lambda_l)$. Since the rank of D_l is m for each l, there exists a matrix $\tilde{A}_l \in \mathrm{GL}_m(\Lambda_l)$ such that

$$D_l\tilde{A}_l = \begin{bmatrix} C_{l,1} \\ \vdots \\ C_{l,s} \end{bmatrix},$$

where $C_{l,i} \in \mathrm{Mat}_{1\times m}(\Lambda_l)$, and for each $1 \le j \le m$ there exists an i such that $C_{l,i} = e_{l,j}$. An elementary pattern of $D_l\tilde{A}_l$ is a choice of $(i_{l1}, \ldots, i_{lm}) \in \{1, \ldots, s\}^m$ such that $C_{l,i_{lk}} = e_{l,k}$ for each $1 \le k \le m$. We fix an elementary pattern $(i_{l1}, \ldots, i_{lm})$ of $D_l\tilde{A}_l$ for each l. For each matrix $P \in \mathrm{Mat}_{s\times m}(\Lambda_l)$ such that, for each $1 \leq j \leq m$ there exists an i such that the i-th row of P is $e_{l,j}$, we define an elementary pattern of P in the same way. For a matrix in $\mathrm{Mat}_{s\times m}(\Lambda)$, we define the procedures

(1) left-multiplication by a matrix in $\mathrm{GL}_s(E)$,

(2_l) right-multiplication by a matrix in $\mathrm{GL}_m(\Lambda_l) \times \prod_{l'\neq l}\{1\}$.

Set $\tilde{A} := (\tilde{A}_l)_l \in \mathrm{GL}_m(\Lambda)$ and $C_i := (C_{l,i})_l \in \mathrm{Mat}_{1\times m}(\Lambda)$. By using the above procedures, we want to transform $D\tilde{A}$ to a matrix $D' = (D'_l)_l$ such that, we can choose an elementary pattern of D'_l to $(1, \dots, m)$ for each l.

Fix $i' \neq i''$ and l_0. Let $\tau = (i'\ i'')$ be the transposition of i' and i''. It is enough to show that, by using the procedures (1) and (2_l), we can transform $D\tilde{A}$ to a matrix $D' = (D'_l)_l$ such that, we can choose an elementary pattern of D'_{l_0} to $(\tau i_{l_0 1}, \dots, \tau i_{l_0 m})$ and an elementary pattern of D'_l to $(i_{l1}, \dots, i_{lm})$ for each $l \neq l_0$.

First we assume that $i' = i_{l_0 j'}$ and $i'' = i_{l_0 j''}$ for some $j' \neq j''$. For $c \in E^\times$, we can exchange the i'-th row of $D\tilde{A}$ for $C_{i'} + cC_{i''}$ by the procedure (1). Since $\#E > d'$, we can take c such that, for each $l \neq l_0$, if $i' = i_{lj}$ for some j then the j-th component of $C_{l,i'} + cC_{l,i''}$ is non-zero. Then by the procedures (2_l) for $l \neq l_0$, we can transform this matrix to a matrix $D'' = (D''_l)_l$ such that, we can choose an elementary pattern of D''_l to $(i_{l1}, \dots, i_{lm})$ for each $l \neq l_0$, the i-th row of D''_{l_0} is $C_{l_0,i}$ for each $i \neq i'$ and the i'-th row of D''_{l_0} is

$$(0, \dots, 0, \overset{\overset{j'}{\vee}}{1}, 0, \dots, 0, \overset{\overset{j''}{\vee}}{c}, 0, \dots, 0).$$

Therefore by the procedure (2_{l_0}), we can transform D'' to a matrix D' which has the desired properties. The case that $i' \notin \{i_{l_0 1}, \dots, i_{l_0 m}\}$ and $i'' = i_{l_0 j''}$ for some j'' is proved in a similar way, and we omit the proof. □

Lemma 5.9. *Assume that $\Gamma(F)$ is Zariski dense in Γ or Λ_l/F is a regular extension for each l. Assume also that $\#E > d'$. We take $1 \leq m \leq s$ and $D \in \mathrm{Mat}_{s\times m}(\Lambda)$ such that $[*, D] \in \mathrm{GL}_s(\Lambda)$ for some $* \in \mathrm{Mat}_{s\times(s-m)}(\Lambda)$. We set*

$$W := \{\mathbf{x} \in \mathrm{Mat}_{1\times s}(\Lambda') | \mathbf{x}D = 0\},$$

and assume that $\Gamma(F')W \subset W$, where the elements of $\Gamma(F')$ act on W by componentwise. Then there exists a matrix $C \in \mathrm{Mat}_{(s-m)\times s}(E)$ such that the rank of C is $s - m$ and $CD = 0$.

Proof. By Lemma 5.8, there exist matrices $A \in \mathrm{GL}_m(\Lambda)$ and $B \in \mathrm{GL}_s(E)$ such that

$$BDA = \begin{bmatrix} I_m \\ C_0 \end{bmatrix},$$

where I_m is the identity matrix of size m and $C_0 \in \mathrm{Mat}_{(s-m)\times m}(\Lambda)$. We set

$$W_B := WB^{-1} = \{\mathbf{x} \in \mathrm{Mat}_{1\times s}(\Lambda') | \mathbf{x}BD = 0\} = \{\mathbf{x} \in \mathrm{Mat}_{1\times s}(\Lambda') | \mathbf{x} \begin{bmatrix} I_m \\ C_0 \end{bmatrix} = 0\}.$$

Then it is clear that W_B is also $\Gamma(F')$-stable. Thus, since each row of $\begin{bmatrix} -C_0 & I_{s-m} \end{bmatrix}$ is an element of W_B, each row of $\begin{bmatrix} -\gamma C_0 & I_{s-m} \end{bmatrix}$ is also an element of W_B for any $\gamma \in \Gamma(F')$.

This means that $\gamma C_0 = C_0$ for each $\gamma \in \Gamma(F')$. Therefore $C_0 \in \mathrm{Mat}_{(s-m)\times m}(E)$ by Theorem 4.20. We set $C := \begin{bmatrix} -C_0 & I_{s-m} \end{bmatrix} B$. Then it is clear that this C has the desired properties. □

Proposition 5.10. *Assume that $\Gamma(F)$ is Zariski dense in Γ or Λ_l/F is a regular extension for each l. Assume also that $\#E > d'$. For any $N \in \mathcal{T}_M$ and Γ-subrepresentation $U \subset \xi_M(N)$, there exists a φ-submodule $N' \subset N$ such that $\xi_M(N') = N$.*

Proof. We take $\mathbf{u} \in \mathrm{Mat}_{u\times 1}(U)$ an F-basis of U such that $\bar{\mathbf{n}} := \begin{bmatrix} \mathbf{u} & * \end{bmatrix}^{\mathrm{tr}}$ forms an F-basis of $\xi_M(N)$. By Lemma 5.2, we have $\bar{\mathbf{n}} = H\mathbf{n}$ for some $H = H(\Psi) \in \mathrm{GL}_s(\Sigma)$. We take a matrix $D \in \mathrm{Mat}_{s\times(s-u)}(\Sigma)$ such that $H^{-1} = \begin{bmatrix} * & D \end{bmatrix}$, and set $W := \{\mathbf{x} \in \mathrm{Mat}_{1\times s}(\Lambda') | \mathbf{x}D = 0\}$. Since $I_s = HH^{-1} = \begin{bmatrix} * & HD \end{bmatrix}$, the i-th row of H is an element of W for each $i \leq u$. These form a Λ'-basis of W because the coefficient ring Λ' is a finite product of fields. For each $\gamma \in \Gamma(F')$, we have $\gamma\bar{\mathbf{n}} = (\gamma H)\mathbf{n} = (\gamma H)H^{-1}\bar{\mathbf{n}}$. Since U is Γ-stable, the (i,j)-th component of $(\gamma H)H^{-1} = \begin{bmatrix} * & (\gamma H)D \end{bmatrix}$ is zero for each $i \leq u$ and $j > u$. Therefore, W is $\Gamma(F')$-stable. By Lemma 5.9, there exists a matrix $C \in \mathrm{Mat}_{u\times s}(E)$ such that the rank of C is u and $CD = 0$. Then we can take $B \in \mathrm{GL}_s(E)$ such that C forms the top rows of B. Let $\begin{bmatrix} \mathbf{n}' & \mathbf{n}'' \end{bmatrix}^{\mathrm{tr}} := B\mathbf{n}$ where $\mathbf{n}' \in \mathrm{Mat}_{u\times 1}(N)$. Let

$$BH^{-1} = \begin{bmatrix} C \\ * \end{bmatrix} \begin{bmatrix} * & D \end{bmatrix} =: \begin{bmatrix} \Psi' & 0 \\ * & * \end{bmatrix},$$

where $\Psi' \in \mathrm{GL}_u(\Sigma)$. Then we have

$$\begin{aligned} \varphi \begin{bmatrix} \mathbf{n}' \\ \mathbf{n}'' \end{bmatrix} &= \varphi(B\mathbf{n}) = \varphi(BH^{-1}H\mathbf{n}) = \sigma(BH^{-1})\varphi(H\mathbf{n}) = \sigma(BH^{-1})H\mathbf{n} \\ &= \sigma(BH^{-1})HB^{-1} \begin{bmatrix} \mathbf{n}' \\ \mathbf{n}'' \end{bmatrix} =: \begin{bmatrix} \Phi' & 0 \\ * & * \end{bmatrix} \begin{bmatrix} \mathbf{n}' \\ \mathbf{n}'' \end{bmatrix}, \end{aligned}$$

where $\Phi' \in \mathrm{GL}_u(E)$. Hence $N' := \langle \mathbf{n}' \rangle_E \subset N$ is a sub φ-module, and we have $\varphi\mathbf{n}' = \Phi'\mathbf{n}'$. Moreover, we have

$$\sigma \begin{bmatrix} \Psi' & 0 \\ * & * \end{bmatrix} = \sigma(BH^{-1}) = (\sigma(BH^{-1})HB^{-1})(BH^{-1}) = \begin{bmatrix} \Phi' & 0 \\ * & * \end{bmatrix} \begin{bmatrix} \Psi' & 0 \\ * & * \end{bmatrix} = \begin{bmatrix} \Phi'\Psi' & 0 \\ * & * \end{bmatrix}.$$

Therefore, Ψ' is a fundamental matrix for Φ'. Since

$$\begin{bmatrix} \mathbf{n}' \\ \mathbf{n}'' \end{bmatrix} = B\mathbf{n} = BH^{-1}\bar{\mathbf{n}} = \begin{bmatrix} \Psi' & 0 \\ * & * \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ * \end{bmatrix},$$

we have that $\xi_M(N') = \langle (\Psi')^{-1}\mathbf{n}' \rangle_F = \langle \mathbf{u} \rangle_F = U$. □

Theorem 5.11. *Assume that $\Gamma(F)$ is Zariski dense in Γ or Λ_l/F is a regular extension for each l. Assume also that $\#E > d'$. Then the morphism of affine F-schemes $\pi_M : \Gamma \to \Gamma_M$ is an isomorphism. Equivalently, the functor $\xi_M : \mathcal{T}_M \to$ **Rep**(Γ, F) is an equivalence of Tannakian categories.*

Proof. By Propositions 5.7 and 5.10, π_M is faithfully flat ([6], Proposition 2.21). On the other hand, π_M is a closed immersion by Theorem 5.6. Therefore π_M is an isomorphism. □

5.2 v-adic case

In this subsection, we continue to use the notations of the previous subsection and consider the case that $(F, E, L) = (\mathbb{F}_q(t)_v, K(t)_v, K^{\mathrm{sep}}(t)_v)$, where the notations are defined in Subsection 3.3.

The assumption that Λ_l/F is regular for each l is not true in general. For example, assume that $r = 1$, $v = t$ and $\Phi \in K$ such that $\Psi := \Phi^{1/(q-1)} \notin K(t)_v$. Then Ψ is a fundamental matrix for Φ, and it is clear that Z is not absolutely irreducible. Therefore the assumptions are not satisfied. However we expect that this assumption is true for "good" objects.

Hence we consider the other assumption. In the v-adic case, $\Gamma(\mathbb{F}_q(t)_v)$ contains a Galois image. Since the Galois image is large enough, we can conclude that $\Gamma(\mathbb{F}_q(t)_v)$ is Zariski dense in Γ.

Lemma 5.12. *Let G be an algebraic group over a field k and H a subgroup of $G(k)$. We set H^{Zar} the Zariski closure of H in G endowed with the reduced structure. Then H^{Zar} is a subgroup scheme of G and smooth.*

Proof. We denote by $\bar{H}$ the Zariski closure of H in $G(\bar{k})$. By Lemma 4.17, $\bar{H}$ is defined over k. Then it is clear that $(H^{Zar})_{\bar{k}} = \bar{H}$. Thus H^{Zar} is absolutely reduced.

To prove that H^{Zar} is a group scheme, it is enough to show that $\bar{H}$ is a group. For any $a \in G(\bar{k})$, the map $G(\bar{k}) \to G(\bar{k}); g \mapsto ag$ is a homeomorphism. Thus for any $a \in H$, we have $a\bar{H} = \overline{aH} \subset \bar{H}$. Thus for any $b \in \bar{H}$, we have $Hb \subset \bar{H}$. Therefore $\bar{H}b = \overline{Hb} \subset \bar{H}$. Hence we have $\bar{H}\bar{H} \subset \bar{H}$. Since the map $G(\bar{k}) \to G(\bar{k}); g \mapsto g^{-1}$ is a homeomorphism, we have $\bar{H}^{-1} = \overline{H^{-1}} = \bar{H}$. □

Lemma 5.13. *Let G be a topological group and k be a topological field. Let $\rho : G \to \mathrm{GL}_r(k)$ be a continuous k-representation of G. We set $\mathcal{C}_\rho$ the Tannakian subcategory of $\mathbf{Rep}(G, k)$ generated by ρ and $\Gamma_\rho \subset \mathrm{GL}_{r/k}$ its Tannakian Galois group. Then $\rho(G)$ is Zariski dense in Γ_ρ.*

Note that the Tannakian Galois group Γ_ρ may not be reduced.

Proof. We have an inclusion $\rho(G)^{Zar} \subset \Gamma_\rho$ and ρ factors through a $\rho(G)^{Zar}(k)$:

$$\rho : G \to \rho(G)^{Zar}(k) \hookrightarrow \Gamma_\rho(k) \hookrightarrow \mathrm{GL}_r(k).$$

Thus we have functors of Tannakian categories

$$\mathcal{C}_\rho \cong \mathbf{Rep}(\Gamma_\rho, k) \to \mathbf{Rep}(\rho(G)^{Zar}, k) \to \mathbf{Rep}(G, k).$$

We denote by $\Gamma_{G,k}$ be the Tannakian Galois group of $\mathbf{Rep}(G, k)$. Then we have morphisms of algebraic groups which correspond to the above sequence:

$$\Gamma_{G,k} \to \rho(G)^{Zar} \hookrightarrow \Gamma_\rho.$$

Since $\Gamma_{G,k} \to \Gamma_\rho$ is an epimorphism of algebraic groups, we have $\rho(G)^{Zar}(\bar{k}) = \Gamma_\rho(\bar{k})$. □

For any $\tau \in G_K$, since $\sigma(\tau\Psi) = \tau(\sigma\Psi) = \tau(\Phi\Psi) = \Phi(\tau\Psi)$, there exists a matrix $A_\tau \in \mathrm{GL}_r(\mathbb{F}_q(t)_v)$ such that $\tau\Psi = \Psi A_\tau$. Therefore we have $\tau(\Sigma) = \Sigma$ and a map $G_K \to \mathrm{Aut}_\sigma(\Sigma/K(t)_v)$. By Lemma 4.16, we have that $A_\tau \in \Gamma(\mathbb{F}_q(t)_v)$ and A_τ corresponds to the image of τ in $\mathrm{Aut}_\sigma(\Sigma/K(t)_v)$ via the isomorphism $\mathrm{Aut}_\sigma(\Sigma/K(t)_v) \cong \Gamma(\mathbb{F}_q(t)_v)$.

On the other hand, we can verify that the map

$$G_K \to \mathrm{Aut}_\sigma(\Sigma/K(t)_v) \cong \Gamma(\mathbb{F}_q(t)_v) \hookrightarrow \Gamma_M(\mathbb{F}_q(t)_v) \hookrightarrow \mathrm{GL}(V(M))$$

coincide with the natural representation $G_K \to \mathrm{GL}(V(M))$ defined in Subsection 3.3.

Proposition 5.14. *The image of G_K in $\Gamma_M(\mathbb{F}_q(t)_v)$ is Zariski dense in Γ_M.*

Proof. Let $\mathcal{C}_M$ be the Tannakian subcategory of $\mathbf{Rep}(G_K, \mathbb{F}_q(t)_v)$ generated by $V(M)$. Then by Theorem 3.26, the categories $\mathcal{T}_M$ and $\mathcal{C}_M$ are equivalence. Therefore Γ_M is also a Tannakian Galois group of $\mathcal{C}_M$. Hence by Lemma 5.13, the image of G_K is Zariski dense in Γ_M. □

Theorem 5.15. *If $(F, E, L) = (\mathbb{F}_q(t)_v, K(t)_v, K^{\mathrm{sep}}(t)_v)$, then the morphism $\pi_M : \Gamma \to \Gamma_M$ is an isomorphism.*

Proof. By Proposition 5.14, $\Gamma(\mathbb{F}_q(t)_v)$ is Zariski dense in Γ_M. In particular, it is Zariski dense in Γ. Therefore by Theorem 5.11, π_M is an isomorphism. □

Proposition 5.16. *Fix an index $m \in \mathbb{Z}/d$ and take an element $\tau \in G_K$ such that $\tau|_{\mathbb{F}_{q^d}} = \sigma|_{\mathbb{F}_{q^d}}^{-m}$. Then the image of τ in $\Gamma(\mathbb{F}_q(t)_v)$ is contained in $\Gamma_m(\mathbb{F}_q(t)_v)$.*

Proof. Since τ induces a $K(t)_v$-isomorphism $K^{\mathrm{sep}}((t-\lambda_{l+m})) \to K^{\mathrm{sep}}((t-\lambda_l))$, also induces a bijection $Z_{l+m}(K^{\mathrm{sep}}((t-\lambda_{l+m}))) \to Z_{l+m}(K^{\mathrm{sep}}((t-\lambda_l)))$. Let $A_\tau \in \Gamma(\mathbb{F}_q(t)_v)$ be as above. Since $\Psi A_\tau = \tau\Psi$, we have $\Psi_l A_\tau = \tau\Psi_{l+m} \in Z_{l+m}(K^{\mathrm{sep}}((t-\lambda_l)))$. Note that $\Psi_l \in Z_l(K^{\mathrm{sep}}((t-\lambda_l)))$ for each l by the definition of Z_l. Therefore by Theorem 4.11, we have $A_\tau \in \Gamma_m(K^{\mathrm{sep}}((t-\lambda_l))) \cap \Gamma(\mathbb{F}_q(t)_v) = \Gamma_m(\mathbb{F}_q(t)_v)$. □

6 v-adic criterion

In this section, we set $K := \mathbb{F}_q(\theta)$ the rational function field over $\mathbb{F}_q$ with one variable θ independent of t. Let M be a finite-dimensional φ-module over $K(t)_v$, $\mathbf{m}$ a $K(t)_v$-basis of M and $\Phi \in \mathrm{Mat}_{r\times r}(K(t)_v)$ a matrix such that $\varphi\mathbf{m} = \Phi\mathbf{m}$.

Definition 6.1. A φ-module M is said to be a *v-adic t-motive* if $\Phi \in \mathrm{Mat}_{r\times r}(K[t]_v)$ and $\det\Phi = c(t-\theta)^s$ for some $c \in \bar{K}^\times$ and $s \in \mathbb{N}$.

Since $t-\theta$ is invertible in $K[t]_v$, v-adic t-motives are $K^{\mathrm{sep}}(t)_v$-trivial by Theorem 3.26. Thus we can apply the results of the previous sections to v-adic t-motives.

Remark 6.2. Let k be a field of characteristic $p > 0$ and $\iota : \mathbb{F}_p[t] \to k$ a ring homomorphism. Anderson defined the notion of t-motives over k in [1]. This is a φ-module M over $k[t]$ which satisfies the following conditions:

- M is free of finite rank over $k[t]$.
- $(t-\iota(t))^N(M/k[t]\cdot\varphi M) = 0$ for some integer $N > 0$.
- M is finitely generated over $k_\sigma[\varphi]$.

Here the φ action on $k[t]$ is defined as before and $k_\sigma[\varphi]$ is the subring of $k[t]_\sigma[\varphi]$ generated by k and φ. Thus we have a functor from the category of t-motives over K (here we take $\iota(t) = \theta$) to the category of v-adic t-motives by tensoring $K(t)_v$.

Let $K_{v(\theta)}$ be the completion of K with respect to the place at $v(\theta)$, $K_d := K \cdot \mathbb{F}_{q^d}$ the composite field of K and $\mathbb{F}_{q^d}$ in $\bar{K}$, $K_{\lambda_l} := K_{d,(\theta-\lambda_l)}$ the completion of K_d with respect to the place at $(\theta - \lambda_l)$, $\overline{K_{\lambda_l}}$ an algebraic closure of K_{λ_l}, and $\mathbb{C}_{\lambda_l} := \widehat{\overline{K_{\lambda_l}}}$ the completion of $\overline{K_{\lambda_l}}$ with respect to the canonical extension of $(\theta - \lambda_l)$. Let v_l be the valuation on $\mathbb{C}_{\lambda_l}$ normalized by $v_l(\theta - \lambda_l) = 1$. For each l, we fix an embedding $\bar{K}$ to $\overline{K_{\lambda_l}}$ over K_d. Then for each $f \in K^{\mathrm{sep}}(t)_v = \prod_l K^{\mathrm{sep}}(\!(t-\lambda_l)\!)$, we can define $f(\theta) \in \prod_l \mathbb{C}_{\lambda_l}$ by substituting θ for t if it converge. We have the following conjecture, which is a v-adic analogue of Proposition 3.1.1 in [2]:

Conjecture 6.3. *Let $\Phi \in \mathrm{GL}_r(K(t)_v) \cap \mathrm{Mat}_{r\times r}(K[t])$ and $\psi \in \mathrm{Mat}_{r\times 1}(K^{\mathrm{sep}}[t]_v)$ be matrices such that $\psi(\theta)$ converges, $\sigma\psi = \Phi\psi$ and $\det \Phi = c(t-\theta)^s$ for some $c \in K^\times$ and $s \in \mathbb{N}$. Then, any linear relation of the components of $\psi(\theta)$ over $K_{v(\theta)}$ lifts to some linear relation of the components of ψ over $K[t]_v$. Precisely speaking, if there exists an element $\rho \in \mathrm{Mat}_{1\times r}(K_{v(\theta)})$ such that $\rho\psi(\theta) = 0$, then there exists an element $P \in \mathrm{Mat}_{1\times r}(K[t]_v)$ such that $P\psi = 0$, $P(\theta)$ converges and $P(\theta) = \rho$.*

Conjecture 6.3 is true if $r = 1$ and we give a proof below. This proof is the same as the proof of the ∞-adic version for $r = 1$ in [2].

If $\rho = 0$, then we can take $P = 0$. Therefore we may assume that $\rho \neq 0$. Since for some $P \in K[t]_v$, we have $P(\theta) = \rho$. Hence it is enough to show that, if $\psi(\theta) = 0$, then $\psi = 0$. Write $\psi = (\sum_i a_{l,i}(t-\lambda_l)^i)_l$. For any $\nu \geq 0$, the infinite sum $\sum_i a_{l,i}((\theta-\lambda_l)^{q^{d\nu}})^i$ converge because $\sum_i a_{l,i}(\theta-\lambda_l)^i$ converges and $v_l((\theta-\lambda_l)^{q^{d\nu}}) \geq v_l(\theta-\lambda_l)$ for each l. Thus we have

$$\psi(\theta^{q^{d\nu}})^{q^d} = (\sum_i a_{l,i}((\theta-\lambda_l)^{q^{d\nu}})^i)_l^{q^d} = (\sum_i a_{l,i}^{q^d}(\theta^{q^{d(\nu+1)}} - \lambda_l)^i)_l = (\sigma^d\psi)(\theta^{q^{d(\nu+1)}}).$$

On the other hand, we have

$$\begin{aligned}(\sigma^d\psi)(\theta^{q^{d(\nu+1)}}) &= (\sigma^{d-1}\Phi)(\theta^{q^{d(\nu+1)}}) \times \cdots \times (\sigma^0\Phi)(\theta^{q^{d(\nu+1)}}) \times \psi(\theta^{q^{d(\nu+1)}}) \\ &= c^{q^{d-1}+\cdots+q^0}(\theta^{q^{d(\nu+1)}} - \theta^{q^{d-1}})^s \cdots (\theta^{q^{d(\nu+1)}} - \theta^{q^0})^s \psi(\theta^{q^{d(\nu+1)}}).\end{aligned}$$

By induction on ν, we have $\sum_i a_{l,i}((\theta-\lambda_l)^{q^{d\nu}})^i = 0$ for each l. Thus the formal series $\sum_i a_{l,i}z^i$ has infinite zeros on the disk $v_l(z) \geq v_l(\theta-\lambda_l)$. Therefore $a_{l,i} = 0$ for all l and i, and we conclude that $\psi = 0$.

Next, we calculate valuations of the coefficients of periods for some examples of t-motives. An element $L_{\alpha,n}$ is an analogue of the n-th Carlitz polylogarithm, and an element Ω_v is an analogue of the Carlitz period.

Proposition 6.4. *Let $n \geq 1$ be an integer and $\alpha \in (K^{\mathrm{sep}})^\times$ an element such that $v_l(\alpha) \geq 0$ for all l. Then there exists an element $L_{\alpha,n} = L_{\alpha,n}(t) = (\sum_i a_{l,i}(t-\lambda_l))_l \in K^{\mathrm{sep}}[t]_v = \prod_l K^{\mathrm{sep}}[\![t-\lambda_l]\!]$ which satisfies the equation*

$$\sigma(L_{\alpha,n}) = \sigma(\alpha) + L_{\alpha,n}/(t-\theta)^n.$$

For any $l \in \mathbb{Z}/d$, $0 \leq m \leq d-1$ and $i \geq 0$, we have

$$v_l(a_{l+m,i}) \geq -q^m\left(\frac{i}{q^d} + \frac{n}{q^d-1}\right).$$

Proof. For an element $L_{\alpha,n} = (\sum_i a_{l,i}(t-\lambda_l))_l \in \prod_l K^{\mathrm{sep}}((t-\lambda_l))$, we have an explicit descriptions

$$(t-\theta)^n \sigma(L_{\alpha,n}) = \left(\sum_i \left(\sum_{j=0}^{n} \binom{n}{j} (\lambda_l - \theta)^{n-j} a_{l-1,i-j}^q \right) (t-\lambda_l)^i \right)_l ,$$

$$\sigma(\alpha)(t-\theta)^n = \left(\sum_{i=0}^{n} \binom{n}{i} (\lambda_l - \theta)^{n-i} \alpha^q (t-\lambda_l)^i \right)_l .$$

We set $b_{l,i} := \sum_{j=1}^{n} \binom{n}{j} (\lambda_l - \theta)^{n-j} a_{l-1,i-j}^q - \binom{n}{i} (\lambda_l - \theta)^{n-i} \alpha^q$ and $c_l := (\lambda_l - \theta)^n$. Then the equation in Proposition is equivalent to the equations

$$a_{l+1,i} = c_{l+1} a_{l,i}^q + b_{l+1,i}$$

for all $l \in \mathbb{Z}/d$ and $i \in \mathbb{Z}$. For $i < 0$, we can take $a_{l,i} = 0$. Fix $i \geq 0$ and consider the system of polynomial equations

$$X_{l+1} = c_{l+1} X_l^q + b_{l+1,i} \quad (l \in \mathbb{Z}/d).$$

For $2 \leq r \leq m$, we set

$$\beta_{m,r,i} := b_{r,i}^{q^{m-r}} \prod_{s=r+1}^{m} c_s^{q^{m-s}} \text{ and } \gamma_m := \prod_{s=2}^{m} c_s^{q^{m-s}}.$$

Then the above equations are equivalent to the equations

$$X_m = \gamma_m X_1^{q^{m-1}} + \sum_{r=2}^{m} \beta_{m,r,i} \quad (2 \leq m \leq d+1).$$

Since $X_{d+1} = X_1$, we can solve these equations in K^{sep}. This proved the existence part of this proposition.

Next we calculate the valuations of these solutions by induction on i. We set $f_i(X_1) := \gamma_{d+1} X_1^{q^d} - X_1 + \sum_{r=2}^{d+1} \beta_{d+1,r,i}$. Since $a_{l,i} = 0$ for all $i < 0$, the inequality for the valuations in the statement of this proposition is true for $i < 0$. Fix $i \geq 0$ and assume that the inequality in the statement of this proposition is true for integers lower than i. It is clear that $v_1(\gamma_{d+1}) = v_1(c_1) = n$. For $2 \leq r \leq d$, we have

$$\begin{aligned} v_1(\beta_{d+1,r,i}) &= n + q^{d+1-r} v_1(b_{r,i}) \\ &\geq n + q^{d+1-r} \min_{1 \leq j \leq n,i} \{ v_1(\binom{n}{j}) + q v_1(a_{r-1,i-j}), v_1(\binom{n}{i}) + q v_1(\alpha) \} \\ &\geq n + q^{d+1-r} \min_{1 \leq j \leq n,i} \{ -q^{r-1} \left(\frac{i-j}{q^d} + \frac{n}{q^d - 1} \right), 0 \} \\ &\geq n + q^{d+1-r} \left(-q^{r-1} \left(\frac{i-1}{q^d} + \frac{n}{q^d - 1} \right) \right) \\ &= n - i + 1 - \frac{q^d n}{q^d - 1}. \end{aligned}$$

For $r = d+1$, we have

$$\begin{aligned} v_1(\beta_{d+1,d+1,i}) &= v_1(b_{1,i}) \\ &\geq \min_{1\leq j\leq n,i}\{n-j+qv_1(a_{d,i-j}), n-i+qv_1(\alpha)+v_1(\binom{n}{i})\} \\ &\geq \min_{1\leq j\leq n,i}\{n-j-q^d\left(\frac{i-j}{q^d}+\frac{n}{q^d-1}\right),0\} \\ &\geq n-i-\frac{q^d n}{q^d-1}. \end{aligned}$$

Thus we conclude that $v_1(\sum_{r=2}^{d+1}\beta_{d+1,r,i}) \geq n-i-q^d n/(q^d-1)$. By considering the Newton polygon of f_i, we have $v_1(a_{1,i}) \geq -i/q^d - n/(q^d-1)$ for any root $a_{1,i}$ of f_i. For $2 \leq r \leq m \leq d$, we have

$$v_1(\beta_{m,r,i}) = q^{m-r}v_1(b_{r,i}) \geq q^{m-r}\left(-q^{r-1}\left(\frac{i-1}{q^d}+\frac{n}{q^d-1}\right)\right) = -q^{m-1}\left(\frac{i-1}{q^d}+\frac{n}{q^d-1}\right)$$

and

$$v_1(\gamma_m a_{1,i}^{q^{m-1}}) = q^{m-1}v_1(a_{1,i}) \geq -q^{m-1}\left(\frac{i}{q^d}+\frac{n}{q^d-1}\right).$$

Thus we have

$$v_1(a_{m,i}) = v_1(\gamma_m a_{1,i}^{q^{m-1}} + \sum_{r=2}^{m}\beta_{m,r,i}) \geq -q^{m-1}\left(\frac{i}{q^d}+\frac{n}{q^d-1}\right).$$

□

The next proposition is proved by similar arguments as Proposition 6.4.

Proposition 6.5. *There exists an element* $\Omega_v = \Omega_v(t) = (\sum_i a_{l,i}(t-\lambda_l))_l \in K^{\mathrm{sep}}[t]_v^\times = \prod_l K^{\mathrm{sep}}[\![t-\lambda_l]\!]^\times$ *which satisfies the equation*

$$\sigma(\Omega_v) = (t-\theta)\Omega_v. \tag{6.1}$$

For any $l \in \mathbb{Z}/d$, $0 \leq m \leq d-1$ *and* $i \geq 0$, *we have*

$$v_l(a_{l+m,i}) = \frac{q^m}{q^{id}(q^d-1)}.$$

By Propositions 6.4 and 6.5, the infinite sums $L_{\alpha,n}(\theta)$ and $\Omega_v(\theta)$ converge.

Example 6.6. We define the *Carlitz motive* to be the φ-module C whose underlying $K(t)_v$-vector space is $K(t)_v$ and on which φ acts by

$$\varphi(f) = (t-\theta)\sigma(f)$$

for each $f \in C$. The equation (6.1) means that the element Ω_v in Proposition 6.5 is a period of C. If we write $\Omega_v = (\Omega_{v,l})_l = (\sum_i a_{l,i}(t-\lambda_l))_l$, then $[K_d(a_{l,0}, a_{l,1}, \dots) : K_d] = \infty$ by Proposition 6.5. Thus $\Omega_{v,l}$ is transcendental over $K(t)_v = K_d(\!(t-\lambda_l)\!)$. Therefore we have that $\mathrm{tr.deg}_{K(t)_v}\Lambda_l = 1$ and $\Gamma_C = \mathbb{G}_m$.

7 Algebraic independence of formal polylogarithms

In this section, we prove the algebraic independence of certain "formal" polylogarithms. The proof of this theorem follows [5] and [10] closely. Let (F, E, L) be a σ-admissible triple and $t, \theta \in E$ distinct elements. Let n, r be positive integers and $\alpha_1, \dots, \alpha_r \in E$ fixed elements. Assume that $(F^\times)_{\mathrm{tor}} \neq F^\times$, and there exist elements $\Omega = (\Omega_l)_l \in L^\times$ and $L_{\alpha_j,n} = (L_{\alpha_j,n,l})_l \in L$ for each $j = 1, \dots, r$ such that $\sigma(\Omega) = (t-\theta)\Omega$, Ω_l is transcendental over E and $\sigma(L_{\alpha_j,n}) = \sigma(\alpha_j) + L_{\alpha_j,n}/(t-\theta)^n$. In the v-adic settings, such elements actually exist if $\alpha_1, \dots, \alpha_r \in K^\times$ (cf. Section 6). We set

$$\Phi := \begin{bmatrix} (t-\theta)^n & & & \\ \sigma(\alpha_1)(t-\theta)^n & 1 & & \\ \vdots & & \ddots & \\ \sigma(\alpha_r)(t-\theta)^n & & & 1 \end{bmatrix} \text{ and } \Psi := \begin{bmatrix} \Omega^n & & & \\ \Omega^n L_{\alpha_1,n} & 1 & & \\ \vdots & & \ddots & \\ \Omega^n L_{\alpha_r,n} & & & 1 \end{bmatrix}.$$

Then we have $\sigma\Psi = \Phi\Psi$. Therefore, if M is the φ-module over E corresponding to Φ, then M is L-trivial. This type of t-motive is considered in [5] and [10]. Note that in ∞-adic case, Ω and $L_{\alpha,n}$ are constructed explicitly, and $L_{\alpha,n}(\theta)$ is the n-th Carlitz polylogarithm of α. We define $\Gamma, \Gamma_M, Z, \Lambda_l, \dots$ as in the previous sections for M, Φ and Ψ. In particular, we have

$$\Lambda_l = E(\Omega_l^n, L_{\alpha,n,l}, \dots, L_{\alpha,n,l}).$$

Furthermore, we assume that, $\Gamma(F)$ is Zariski dense in Γ or Λ_l/F is regular extension for each l. Thus the natural immersion $\Gamma \to \Gamma_M$ is an isomorphism by Theorem 5.11.

For each F-algebra R, we set

$$G(R) := \left\{ \begin{bmatrix} * & 0 & \cdots & 0 \\ * & 1 & & \\ \vdots & & \ddots & \\ * & & & 1 \end{bmatrix} \in \mathrm{GL}_{r+1}(R) \right\}.$$

Then G is an algebraic group over F and we have a natural inclusion $\Gamma \subset G$. Let $X_0, \dots, X_r$ be the coordinates of G such that the first column of a general element of G "is"

$$\begin{bmatrix} X_0 & & & \\ X_1 & 1 & & \\ \vdots & & \ddots & \\ X_r & & & 1 \end{bmatrix}.$$

We have the exact sequence $1 \to \mathbb{G}_a^r \to G \to \mathbb{G}_m \to 1$, here $\mathbb{G}_a^r$ is the subgroup scheme of G with coordinates $(X_1, \dots, X_r)$ and $\mathbb{G}_m$ is the quotient of G given by the projection $(X_i) \mapsto X_0$. Let $C \in \Phi\mathrm{M}_E^L$ be the one-dimensional φ-module such that $\varphi(f) = (t-\theta)\sigma(f)$ for each $f \in C = E$. Then we have the following exact sequence:

$$0 \to C^{\otimes n} \to M \to \mathbf{1}^r \to 0.$$

Thus $C^{\otimes n}$ is an object of $\mathcal{T}_M$ and we have the canonical surjection $\pi \colon \Gamma \cong \Gamma_M \to \Gamma_{C^{\otimes n}} \cong \mathbb{G}_m$. We set $V := \ker \pi$. Then we have the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & V & \longrightarrow & \Gamma & \xrightarrow{\pi} & \mathbb{G}_m & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \mathbb{G}_a^r & \longrightarrow & G & \longrightarrow & \mathbb{G}_m & \longrightarrow & 1, \end{array}$$

where the rows are exact.

Proposition 7.1. *The subgroup V of $\mathbb{G}_a^r$ is defined by linear forms in $X_1, \dots, X_r$ with F coefficients.*

Proof. Let $T \subset \Gamma_{\bar{F}}$ be a maximal torus and $\bar{\pi}\colon \Gamma_{\bar{F}} \to \mathbb{G}_{m,\bar{F}}$ be the base extension of π to $\bar{F}$. Then we have $\dim T = 1$ and $\bar{\pi}|_T\colon T \to \mathbb{G}_{m,\bar{F}}$ is an isomorphism. Thus $d\bar{\pi}$ is non-trivial and so is $d\pi$. Hence we have the following exact sequence:

$$0 \longrightarrow \operatorname{Lie} V \longrightarrow \operatorname{Lie} \Gamma \longrightarrow \operatorname{Lie} \mathbb{G}_m \longrightarrow 0.$$

Since Γ and $\mathbb{G}_m$ are smooth over F, we have the equalities $\dim_F \operatorname{Lie}\Gamma = \dim \Gamma$ and $\dim_F \operatorname{Lie}\mathbb{G}_m = 1$. Thus we have the equality $\dim_F \operatorname{Lie} V = \dim V$. Therefore V is smooth over F. Thus it is enough to show that the space $V(\bar{F})$ is a linear space defined over F. Let

$$\mu = \begin{bmatrix} 1 & 0 \\ v & I_r \end{bmatrix} \in V(\bar{F}) \text{ and } \alpha \in \bar{F}^\times$$

be any elements. Since $\Gamma(\bar{F}) \to \mathbb{G}_m(\bar{F})$ is surjective, there exists an element $\gamma \in \Gamma(\bar{F})$ such that $\pi(\gamma) = \alpha$. Then we have

$$V(\bar{F}) \ni \gamma^{-1}\mu\gamma = \begin{bmatrix} 1 & 0 \\ \alpha v & I_r \end{bmatrix}.$$

Thus $V(\bar{F})$ is a linear subspace of $\mathbb{G}_a^r(\bar{F})$. Since V is defined over F, V is defined by linear forms in $X_1, \dots, X_r$ with F coefficients. □

Since V is smooth and $\mathrm{H}^1(F, V) = 1$, we have the exact sequence

$$1 \longrightarrow V(F) \longrightarrow \Gamma(F) \longrightarrow \mathbb{G}_m(F) \longrightarrow 1.$$

By the assumption on F, there exists an element $b_0 \in F^\times \smallsetminus (F^\times)_{\mathrm{tor}}$. By the above sequence, there exists an element

$$\gamma = \begin{bmatrix} b_0 & & & \\ b_1 & 1 & & \\ \vdots & & \ddots & \\ b_r & & & 1 \end{bmatrix} \in \Gamma(F).$$

We fix such b_0 and γ. For each F-algebra R and $a \in R^\times$, we set

$$\gamma_a := \begin{bmatrix} a & & & \\ \frac{b_1}{b_0 - 1}(a-1) & 1 & & \\ \vdots & & \ddots & \\ \frac{b_r}{b_0-1}(a-1) & & & 1 \end{bmatrix}.$$

Then for each $a, b \in R^\times$ and $m \in \mathbb{Z}$, we have $\gamma_a\gamma_b = \gamma_{ab}$ and $\gamma^m = \gamma_{b_0^m}$. Hence we have $\overline{\langle\gamma\rangle} = (R \mapsto \{\gamma_a | a \in R^\times\})$, a line in Γ. We set $\Gamma' := \overline{\langle V, \gamma\rangle} \subset \Gamma$ and $s := r - \dim V$. We claim that $\Gamma' = \Gamma$. Indeed, let

$$F_i = \sum_{j=1}^{r} c_{i,j} X_j \in F[X_1, \dots, X_r] \quad (i = 1, \dots, s) \tag{7.1}$$

be linear forms defining V. For each i, we set

$$G_i := (b_0 - 1)F_i(X_1, \dots, X_r) - F_i(b_1, \dots, b_r)(X_0 - 1) \in F[X_0, \dots, X_r].$$

Then we can verify that $G_1, \dots, G_s$ define Γ' in GL_{r+1} and Γ' is an algebraic group. Since $V \subset \Gamma' \subset \Gamma$ and $\Gamma' \to \mathbb{G}_m$ is surjective, we have $\Gamma' = \Gamma$. Thus we have the following proposition:

Proposition 7.2. *The algebraic group Γ is defined by the linear polynomials $G_1, \dots, G_s$ in $\mathrm{GL}_{r+1/F}$.*

Since $Z_{\bar{E}} \cong \Gamma_{\bar{E}}$ and Z is defined over E, Z is defined by linear polynomials over E, and there exists an E-valued point

$$\xi = \begin{bmatrix} f_0 & & & \\ f_1 & 1 & & \\ \vdots & & \ddots & \\ f_r & & & 1 \end{bmatrix} \in Z(E).$$

We fix such ξ. Then we have $Z = \xi \cdot \Gamma_E$. Set $f'_i := G_i(f_0, \dots, f_r)f_0^{-1} \in E$ and $H_i := G_i(X_0, \dots, X_r) - X_0 f'_i \in E[X_0, \dots, X_r]$. Then $H_1, \dots, H_s$ are defining polynomials for Z. If we set $g_i := \sum_{j=1}^{r} c_{i,j} b_j$, then we have

$$H_i = (b_0 - 1)\sum_{j=1}^{r} c_{i,j} X_j + g_i - (g_i + f'_i)X_0.$$

Since $\Psi_l \in Z(\Sigma_l)$ for each l, we have

$$(b_0 - 1)\sum_{j=1}^{r} c_{i,j} L_{\alpha_j,n,l} + g_i \Omega_l^{-n} - (g_i + f'_i) = 0$$

for each l and i. Set $B := (c_{i,j})_{i,j} \in \mathrm{Mat}_{s \times r}(F)$. By the definition of $c_{i,j}$ (7.1), the rank of B is $s = r - \dim V$. Set

$$P = \begin{bmatrix} P_1 \\ \vdots \\ P_s \end{bmatrix} := \begin{bmatrix} (b_0 - 1)c_{1,1} & \dots & (b_0 - 1)c_{1,r} & g_1 & -(g_1 + f'_1) \\ \vdots & & \vdots & \vdots & \vdots \\ (b_0 - 1)c_{s,1} & \dots & (b_0 - 1)c_{s,r} & g_s & -(g_s + f'_s) \end{bmatrix} \in \mathrm{Mat}_{s \times (r+2)}(E),$$

the coefficients matrix of the above equations. Then the rank of P is also s. We interested in

$$N_l := \langle L_{\alpha_1,n,l}, \dots, L_{\alpha_r,n,l}, \Omega_l^{-n}, 1 \rangle_E \subset \Lambda_l.$$

This is the image of the E-linear map

$$\beta_l \colon E^{r+2} \to \Lambda_l;\ (x_1, \dots, x_{r+2}) \mapsto \sum_{j=1}^{r} x_j L_{\alpha_j,n,l} + x_{r+1}\Omega_l^{-n} + x_{r+2}.$$

Since $P_i \in \ker \beta_l$ for each i, we have the inequality $\dim_E \ker \beta_l \geq s$. Thus we have $\dim_E N_l \leq r + 2 - s = \dim V + 2 = \dim \Gamma + 1 = \mathrm{tr.deg}_E \Lambda_{l'} + 1$ for each l and l'. On the other hand, it is clear that $\dim_E N_l \geq \mathrm{tr.deg}_E \Lambda_l = \mathrm{tr.deg}_E \Lambda_{l'}$. Thus we have the following theorem:

Theorem 7.3. *For each l and l', we have* $\mathrm{tr.deg}_E \Lambda_{l'} \leq \dim_E N_l \leq \mathrm{tr.deg}_E \Lambda_{l'} + 1$.

Corollary 7.4. *If $L_{\alpha_1,n,l}, \dots, L_{\alpha_r,n,l}, 1$ are linearly independent over E for some l, then $L_{\alpha_1,n,l'}, \dots, L_{\alpha_r,n,l'}$ are algebraically independent over E for each l'.*

Proof. Note that since $\Lambda_{l'} = E(\Omega_{l'}^n, L_{\alpha_1,n,l'}, \dots, L_{\alpha_r,n,l'})$, we have $\mathrm{tr.deg}_E \Lambda_{l'} \leq r+1$. By the assumption, we have $r+1 \leq \dim_E N_l \leq r+2$.

Assume that $\dim_E N_l = r+2$. Then $\mathrm{tr.deg}_E \Lambda_{l'} < \dim_E N_l$. By Theorem 7.3, we have $\dim_E N_l = \mathrm{tr.deg}_E \Lambda_{l'} + 1$. Thus we have $\mathrm{tr.deg}_E \Lambda_{l'} = r+1$ and $\Omega_{l'}^n, L_{\alpha_1,n,l'}, \dots, L_{\alpha_r,n,l'}$ are algebraically independent over E.

On the other hand, assume that $\dim_E N_l = r+1$. By the assumption, we can write Ω_l^{-n} as a linear combination of $L_{\alpha_1,n,l}, \dots, L_{\alpha_r,n,l}, 1$ over E. In particular, we have $\Omega_l^n \in E(L_{\alpha_1,n,l}, \dots, L_{\alpha_r,n,l})$. Letting σ act on this relation, we have

$$(t-\theta)^n \Omega_{l+1}^n \in \sigma(E)\left(\sigma(\alpha_1) + \frac{L_{\alpha_1,n,l+1}}{(t-\theta)^n}, \dots, \sigma(\alpha_r) + \frac{L_{\alpha_r,n,l+1}}{(t-\theta)^n})\right).$$

Thus for each l', we have $\Omega_{l'}^n \in E(L_{\alpha_1,n,l'}, \dots, L_{\alpha_r,n,l'})$ and $\Lambda_{l'} = E(L_{\alpha_1,n,l'}, \dots, L_{\alpha_r,n,l'})$. By Theorem 7.3, we have $\mathrm{tr.deg}_E \Lambda_{l'} \geq \dim_E N_l - 1 = r$. Thus $\mathrm{tr.deg}_E \Lambda_{l'} = r$ and $L_{\alpha_1,n,l'}, \dots, L_{\alpha_r,n,l'}$ are algebraically independent over E. □

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