On the Alexander stratification in the deformation space of Galois characters

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Abstract. Following the analogy between primes and knots, we give an arithmetic analogue of a theorem by E. Hironaka on the Alexander stratification in the character variety of a knot group.

Introduction

Based on the analogy between primes and knots, there are close paralells between Iwasawa theory and Alexander-Fox theory ([Mo1∼3]). The purpose of this paper is to add an item to the dictionary of *arithmetic topology* by showing an arithmetic counterpart of a theorem by E. Hironaka ([Hr], see also [L]) on the Alexander stratification in the character variety of a knot group.

Let K be a tame knot in the 3-sphere S^3 and set $X_K := S^3 \setminus K$. Let G_K be the knot group $\pi_1(X_K)$ and let $\mathcal{X}_K$ be the group of homomorphisms $\rho : G_K \to \mathbf{C}^\times$. Since $\rho \in \mathcal{X}_K$ factors through $H_1(X_K, \mathbf{Z})$ which is an infinite cyclic group generated by a meridian α, we have the isomorphism $\mathcal{X}_K \simeq \mathbf{C}^\times$ by sending ρ to $\rho(\alpha)$. Let X_K^∞ be the infinite cyclic cover of X_K with Galois group $\mathrm{Gal}(X_K^\infty/X_K) = \langle\alpha\rangle$. The homology group $H_1(X_K^\infty, \mathbf{Z})$ is regarded as a module over $\mathbf{Z}[\langle\alpha\rangle] = \mathbf{Z}[t, t^{-1}]$ $(\alpha = t)$, the Laurent polynomial ring. For an integer $d \geq 1$, let $J_{K,d-1}$ be the $(d-1)$-th elementary ideal of a $\mathbf{Z}[t, t^{-1}]$-module $H_1(X_K^\infty, \mathbf{Z})$, called the d-th Alexander ideal (cf. [Hl],Ch.3), and we define the *Alexander stratification* in $\mathcal{X}_K$ by $\mathcal{X}_K(d) := \{\rho \in \mathcal{X}_K \,|\, f(\rho(\alpha)) = 0$ for all $f \in J_{K,d-1}\}$: $\mathcal{X}_K \supset \mathcal{X}_K(1) \supset \cdots \supset \mathcal{X}_K(d) \supset \cdots$. On the other hand, we also define the *cohomology-jump* stratification in $\mathcal{X}_K$ by $\mathcal{Y}_K(d) := \{\rho \in \mathcal{X}_K \,|\, \dim H^1(G_K, \mathbf{C}(\rho)) \geq d\}$ where $\mathbf{C}(\rho)$ is the additive group $\mathbf{C}$ equipped with the action of $g \in G_K$ as multiplication by $\rho(g)$. Then E. Hironaka ([Hr]) showed that these stratifications essentially coincide:

$$(0.1) \qquad \mathcal{Y}_K(d) = \mathcal{X}_K(d)\,(d \neq 1),\ \mathcal{Y}_K(1) = \mathcal{X}_K(1) \cup \{\mathbf{1}\},$$

where $\mathbf{1}$ is the trivial representation defined by $\mathbf{1}(g) = 1$ for all $g \in G_K$.

In order to obtain an arithmetic counterpart of (0.1), let us recall a part of the basic analogies in arithmetic topology:

knot $K : S^1 \hookrightarrow S^3$	$\longleftrightarrow$	prime $\mathrm{Spec}(\mathbf{F}_p) \hookrightarrow \mathrm{Spec}(\mathbf{Z})$
$X_K = S^3 \setminus K$	$\longleftrightarrow$	$X_p = \mathrm{Spec}(\mathbf{Z}[1/p])$
$G_K = \pi_1(X_K)$	$\longleftrightarrow$	$G_p = \pi_1^{\acute{e}t}(X_p)$
$X_K^\infty \to X_K$	$\longleftrightarrow$	$X_p^\infty = \mathrm{Spec}(\mathbf{Z}[\mu_{p^\infty}, 1/p]) \to X_p$
$\mathbf{Z}[\mathrm{Gal}(X_K^\infty/X_K)]$ Laurent polynomial ring	$\longleftrightarrow$	$\mathbf{Z}_p[[\mathrm{Gal}(X_p^\infty/X_p)]]$ Iwasawa ring
$H_1(X_K^\infty, \mathbf{Z})$	$\longleftrightarrow$	$H_1(X_p^\infty, \mathbf{Z}_p)$
Alexander ideals $J_{K,d}$ Alexander polynomial	$\longleftrightarrow$	Iwasawa ideals $J_{p,d}$ (see Section 2) Iwasawa polynomial
$\mathcal{X}_K = \mathrm{Hom}(G_K, \mathbf{C}^\times)$	$\longleftrightarrow$	deformation space $\mathcal{X}_p$ of characters of G_p

where μ_{p^∞} is the group of all p^n-th roots of unity for $n \geq 1$.

Based on these analogies, we shall introduce arithmetic analogues of the Alexander and cohomology-jump stratifications and establish their relation, Main Theorem 2.3, which is exactly analogous to the case of a knot. A fact that the zeros of the Iwasawa polynomial give rise to the cohomology-jump divisor was delivered (without proof) in B. Mazur's lecture [Ma3] (see also [Ma2]) and a motivation of our work was to give a proof of this fact and to generalize it for higher Iwasawa ideals, following the analogy with knot theory.

1. The universal deformation space of Galois representations

Thoughout this paper, we fix an odd prime number p and denote by G_p the étale fundamental group of $\mathrm{Spec}(\mathbf{Z}) \setminus \{(p)\}$, namely, the Galois group of the maximal algebraic extension $\mathbf{Q}_{\{p,\infty\}}$ of $\mathbf{Q}$ unramified outside p and the infinite prime ∞.

For an integer $n \geq 1$, let $\bar{\rho} : G_p \to GL_n(\mathbf{F}_p)$ be an absolutely irreducible, continuous representation of G_p over the field $\mathbf{F}_p$ of p elements. Following B. Mazur [Mz1], a *deformation* of $\bar{\rho}$ is then defined to be a pair of a complete noetherian local ring R with residue field $R/m_R = \mathbf{F}_p$ (m_R being the maximal ideal of R) and a strict equivalence class of continuous representations $\rho : G_p \to GL_n(R)$ which are liftings of $\bar{\rho}$, i.e, $\rho \bmod m_R = \bar{\rho}$. Here two liftings of $\bar{\rho}$ to R are said to be strictly equivalent if they are conjugate by an element of the kernel of the reduction map $GL_n(R) \to GL_n(\mathbf{F}_p)$. Recall the following fundamental theorem, due to B. Mazur, in the deformation theory of Galois representations:

Theorem 1.1 ([Ma1],1.2). *There exists a complete noetherian local ring $R(\bar{\rho})$ with residue field $\mathbf{F}_p$ and a continuous lift $\rho_n : G_p \to GL_n(R(\bar{\rho}))$ of $\bar{\rho}$ which satisfy the following universal property: For any lift ρ of $\bar{\rho}$ to a complete noetherian local ring R, there exists uniquely a local ring homomorphism $\varphi : R(\bar{\rho}) \to R$ such that $\varphi \circ \rho$ is strictly equivalent to ρ_n.*

The ring $R(\bar{\rho})$ is called the *universal deformation ring* of $\bar{\rho}$ and the representation ρ_n is called the *universal deformation* of $\bar{\rho}$. We define the *universal deformation space* of $\bar{\rho}$ by

$$\mathcal{X}_p(\bar{\rho}) := \mathrm{Hom}_{\mathbf{Z}_p}(R(\bar{\rho}), \mathbf{C}_p)$$

where $\mathbf{C}_p$ is the completion of an algebraic closure of $\mathbf{Q}_p$ with respect to the p-adic norm $|\cdot|_p$ and $\mathrm{Hom}_{\mathbf{Z}_p}$ stands for the set of $\mathbf{Z}_p$-algebra homomorphisms.

For the case $n = 1$, we have the following description of $\mathcal{X}_p(\bar{\rho})$. Let $\mathbf{Q}_\infty$ denote the subextension of $\mathbf{Q}_{\{p,\infty\}}/\mathbf{Q}$ with Galois group $\Gamma := \mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})$ being isomorphic to $\mathbf{Z}_p$. We write g_p for the image of $g \in G_p$ under the natural map $G_p \to \Gamma$. Let $\tilde{\rho} : G_p \to \mathbf{Z}_p^\times$ be the lift of $\bar{\rho}$ obtained by composing $\bar{\rho}$ with the natural inclusion $\mathbf{F}_p^\times \subset \mathbf{Z}_p^\times$. Then we have

Theorem 1.2 ([Ma1],1.4). *The universal deformation ring $R(\bar{\rho})$ of the character $\bar{\rho} : G_p \to \mathbf{F}_p^\times$ is the completed group algebra $\mathbf{Z}_p[[\Gamma]]$ and the universal deformation $\rho_1 : G_p \to \mathbf{Z}_p[[\Gamma]]^\times$ is given by $\rho_1(g) = \tilde{\rho}(g)g_p$.*

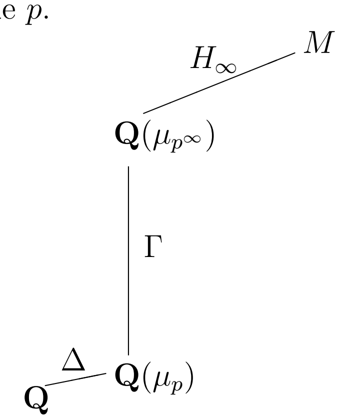
We fix a topological generator γ of Γ once and for all and identify $\mathbf{Z}_p[[\Gamma]]$ with the power series ring $\Lambda := \mathbf{Z}_p[[T]]$ with variable T by sending γ to $1+T$. The universal deformation space $\mathcal{X}_p(\bar{\rho}) = \mathrm{Hom}_{\mathbf{Z}_p}(\Lambda, \mathbf{C}_p)$ is then given by the unit disk $\mathcal{D} := \{x \in \mathbf{C}_p \,|\, |x|_p < 1\}$:

$$\mathcal{X}_p(\bar{\rho}) \simeq \mathcal{D};\ \varphi \mapsto \varphi(T) = \varphi(\gamma) - 1.$$

Remark 1.3. Let F be the subfield of $\mathbf{Q}_{\{p,\infty\}}$ fixed by the kernel of $\bar{\rho} : G_p \to GL_n(\mathbf{F}_p)$. Any deformation of $\bar{\rho}$ then factors through the Galois group over $\mathbf{Q}$ of the maximal pro-p extension of F unramified outside p, in particular, factors through the Galois group G of the maximal pro-p extension of $\mathbf{Q}(\mu_p)$ unramified outside p. Hence, one may define the universal deformation space $\mathcal{X}_p(\bar{\rho})$ using the smaller Galois group G instead of G_p.

2. The Alexander and cohomology-jump stratifications

Let μ_{p^n} be the group of p^n-th roots of unity and set $\mu_{p^\infty} = \cup_{n=1}^\infty \mu_{p^n}$. Let $\kappa : \mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q}) \xrightarrow{\sim} \mathbf{Z}_p^\times$ be the isomorphism defined by $\sigma(\zeta) = \zeta^{\kappa(\sigma)}$ for $\sigma \in \mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q}), \zeta \in \mu_{p^\infty}$. Let Δ be the Galois group $\mathrm{Gal}(\mathbf{Q}(\mu_p)/\mathbf{Q})$ which is isomorphic to $\mathbf{F}_p^\times$. Let M be the maximal abelian pro-p extension of $\mathbf{Q}(\mu_{p^\infty})$ unramified outside p.



The Galois group $\mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q})$ acts naturally on the étale homology group $H_\infty := H_1(\mathrm{Spec}(\mathbf{Z}[\mu_{p^\infty}, p^{-1}]), \mathbf{Z}_p) \simeq \mathrm{Gal}(M/\mathbf{Q}(\mu_{p^\infty}))$. Note that we have the canonical decomposition $\mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q}) = \Delta \times \Gamma$. Let $H_\infty^{(i)}$ be the maximal Λ-submodule of H_∞ on which $\delta \in \Delta$ acts as multiplication by $\kappa(\delta)^i$ so that we have the decomposition

$$H_\infty = \bigoplus_{i \in \mathbf{F}_p^\times} H_\infty^{(i)}.$$

Let $J_{p,d}^{(i)}$ be the d-th elementary ideal of a Λ-module $H_\infty^{(i)}$, called the d-th *Iwasawa ideal* of $H_\infty^{(i)}$. Note that $J_{p,0}^{(i)}$ is generated by the characteristic polynomial $I_p^{(i)}(T) := \det(T \cdot id - (\gamma - 1) \,|\, H_\infty^{(i)} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p)$, the original *Iwasawa polynomial*.

Now, let $\omega : G_p \to \mathbf{F}_p^\times$ be the composition of the natural map $G_p \to \Delta$ with $\Delta \simeq \mathbf{F}_p^\times (\subset \mathbf{Z}_p^\times)$. For i mod $p-1$, let $\rho^{(i)} : G_p \to \Lambda^\times$ be the universal deformation of ω^i, and $\mathcal{X}_p^{(i)}$ the universal deformation space of ω^i as given in Theorem 1.2. In the following, we write $\rho_\varphi^{(i)}$ for $\varphi \circ \rho^{(i)}$ for $\varphi \in \mathcal{X}_p^{(i)}$.

Definition 2.1. For $d \geq 1$ and an integer i mod $p-1$, we define the *Alexander stratification* $\mathcal{X}_p^{(i)}(d)$ in $\mathcal{X}_p^{(i)}$ by

$$\begin{aligned} \mathcal{X}_p^{(i)}(d) &:= \mathrm{Hom}_{\mathbf{Z}_p}(\Lambda/J_{p,d-1}^{(i)}, \mathbf{C}_p) \\ &= \{\varphi \in \mathcal{X}_p^{(i)} \,|\, f(\varphi(\gamma) - 1) = 0 \text{ for all } f \in J_{p,d-1}^{(i)}\}. \end{aligned}$$

On the other hand, let $\mathbf{C}_p(\rho_\varphi^{(i)})$ denote the additive group $\mathbf{C}_p$ on which $g \in G_p$ acts as multiplication by $\rho_\varphi^{(i)}(g)$. We then introduce another stratification in $\mathcal{X}_p^{(i)}$ as follows.

Definition 2.2. For $d \geq 1$, we define the *cohomology-jump stratification* $\mathcal{Y}_p^{(i)}(d)$ in $\mathcal{X}_p^{(i)}$ by

$$\mathcal{Y}_p^{(i)}(d) := \{\varphi \in \mathcal{X}_p^{(i)} \,|\, \dim H^1(G_p, \mathbf{C}_p(\rho_\varphi^{(i)})) \geq d\}.$$

Our main result claims that both stratifications essentially coincide.

Main Theorem 2.3. *For even i mod $p-1$, we have*

$\mathcal{Y}_p^{(i)}(d) = \mathcal{X}_p^{(i)}(d)$ $(d \neq 1)$, $\mathcal{Y}_p^{(i)}(1) = \mathcal{X}_p^{(i)}(1)$ $(i \neq 0)$, $\mathcal{Y}_p^{(0)}(1) = \mathcal{X}_p^{(0)}(1) \cup \{\mathbf{1}\}$

where $\mathbf{1}$ *is the trivial* $\mathbf{Z}_p$*-homomorphism defined by* $\mathbf{1}(\gamma) = 1$.

In particular, we have the following

Corollary 2.4. *For even* $i \neq 0$ *mod* $p-1$*, we have*

$$I_p^{(i)}(\varphi(\gamma)-1) = 0 \Longleftrightarrow \dim H^1(G_p, \mathbf{C}_p(\rho_\varphi^{(i)})) \geq 1.$$

3. The Alexander module and cohomology

We fix some notations used in the sequel: For an integer i mod $p-1$,
$\mathbf{Z}_p^{(i)} = \mathbf{Z}_p$ equipped with the action of $\delta \in \Delta$ as multiplication by $\omega^i(\delta)$.
$\tilde{\Lambda} := \mathbf{Z}_p[[\Delta \times \Gamma]] = \mathbf{Z}_p[\Delta][[\Gamma]]$.
$\Lambda^{(i)} := \mathbf{Z}_p^{(i)}[[\Gamma]] = \Lambda$ equipped with the action of $\delta \in \Delta$ as multiplication by $\omega^i(\delta)$.
$\psi : \mathbf{Z}_p[[G_p]] \to \tilde{\Lambda}$ = the map induced by the natural map $G_p \to \mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q})$.
$K := \mathrm{Ker}(\psi)$. Note that the pro-p abelianization of K is H_∞.
$\epsilon_{\mathbf{Z}_p[[\mathcal{G}]]} :=$ the augmentation map $\mathbf{Z}_p[[\mathcal{G}]] \to \mathbf{Z}_p$ for a profinite group $\mathcal{G}$.
$I_{\mathbf{Z}_p[[\mathcal{G}]]} := \mathrm{Ker}(\epsilon_{\mathbf{Z}_p[[\mathcal{G}]]})$.

Definition 3.1. We define the *Alexander module* A_p of a prime $\mathrm{Spec}(\mathbf{F}_p)$ by the ψ-differential module of G_p over $\tilde{\Lambda}$, namely, the quotient module of the free $\tilde{\Lambda}$-module on symbols dg, $g \in G_p$, by the closed submodule generated by $d(fg) - df - \psi(f)dg$ for all $f, g \in G_p$:

$$A_p := \bigoplus_{g \in G_p} \tilde{\Lambda} dg / \langle d(fg) - df - \psi(f)dg \rangle_{\tilde{\Lambda}}.$$

Lemma 3.2. *We have an isomorphism over* $\tilde{\Lambda}$:

$$A_p \simeq I_{\mathbf{Z}_p[[G_p]]} / I_{\mathbf{Z}_p[[K]]} I_{\mathbf{Z}_p[[G_p]]}$$

where $\lambda \in \tilde{\Lambda}$ acts on the r.h.s. as multiplication by $\alpha \in \psi^{-1}(\lambda)$.

Proof. It is easy to see that the map

$$A_p \to I_{\mathbf{Z}_p[[G_p]]} \otimes_{\mathbf{Z}_p[[G_p]]} \tilde{\Lambda}\ ;\ \ \sum \lambda_g dg \mapsto \sum (g-1) \otimes \lambda_g$$

gives a $\tilde{\Lambda}$-isomorphism. On the other hand, ψ induces a $\mathbf{Z}_p[[G_p]]$-isomorphism $\mathbf{Z}_p[[G_p]]/I_{\mathbf{Z}_p[[K]]}\mathbf{Z}_p[[G_p]] \simeq \tilde{\Lambda}$. Hence we have the isomorphism over $\tilde{\Lambda}$

$$A_p \simeq I_{\mathbf{Z}_p[[G_p]]} \otimes_{\mathbf{Z}_p[[G_p]]} (\mathbf{Z}_p[[G_p]]/I_{\mathbf{Z}_p[[K]]}\mathbf{Z}_p[[G_p]]) \simeq I_{\mathbf{Z}_p[[G_p]]}/I_{\mathbf{Z}_p[[K]]}I_{\mathbf{Z}_p[[G_p]]}. \ \Box$$

We set $A_p^{(i)} := A_p \otimes_{\tilde{\Lambda}} \Lambda^{(i)}$ where $\Lambda^{(i)}$ is regarded as a $\tilde{\Lambda}$-module through the map $\omega^i : \mathbf{Z}_p[\Delta] \to \mathbf{Z}_p^{(i)}$. The next Proposition is regarded as an arithmetic analog of the Crowell exact sequence in knot theory (cf. [Hl],4.1).

Proposition 3.3. *We have an exact sequence of $\tilde{\Lambda}$-modules*

$$0 \longrightarrow H_\infty \longrightarrow A_p \longrightarrow I_{\tilde{\Lambda}} \longrightarrow 0$$

and, for each i mod $p-1$, an exact sequence of Λ-modules

$$0 \longrightarrow H_\infty^{(i)} \longrightarrow A_p^{(i)} \longrightarrow I_{\Lambda^{(i)}} \longrightarrow 0.$$

Proof. We follow the argument given in [NSW],Proposition 5.6.7. Consider the exact sequence

$$(*) \qquad 0 \longrightarrow I_{\mathbf{Z}_p[[G_p]]} \longrightarrow \mathbf{Z}_p[[G_p]] \overset{\epsilon_{\mathbf{Z}_p[[G]]}}{\longrightarrow} \mathbf{Z}_p \longrightarrow 0.$$

Take the K-homology sequence of $(*)$ to get an exact sequence of $\tilde{\Lambda}$-modules

$$H_1(K, \mathbf{Z}_p[[G_p]]) \to H_1(K, \mathbf{Z}_p) \to H_0(K, I_{\mathbf{Z}_p[[G_p]]}) \to H_0(K, \mathbf{Z}_p[[G_p]]) \to \mathbf{Z}_p.$$

Here we have:
$H_1(K, \mathbf{Z}_p[[G_p]]) = 0$ since $\mathbf{Z}_p[[G_p]]$ is an induced K-module,
$H_1(K, \mathbf{Z}_p) = H_\infty$,
$H_0(K, I_{\mathbf{Z}_p[[G_p]]}) = I_{\mathbf{Z}_p[[G_p]]}/I_{\mathbf{Z}_p[[K]]}I_{\mathbf{Z}_p[[G_p]]} = A_p$ by Lemma 3.2,
$H_0(K, \mathbf{Z}_p[[G_p]]) = \mathbf{Z}_p[[G_p]]/I_{\mathbf{Z}_p[[K]]}\mathbf{Z}_p[[G_p]] = \tilde{\Lambda}$
and so we obtain the first exact sequence of $\tilde{\Lambda}$-modules. The latter exact sequence is obtained by tensoring the first one with $\Lambda^{(i)}$ over $\tilde{\Lambda}$. $\Box$

We set $A_p(\rho_\varphi^{(i)}) = A_p \otimes_{\tilde{\Lambda}} \mathbf{C}_p(\rho_\varphi^{(i)})$. If we write $\mathbf{C}_p(\varphi)$ for the additive group $\mathbf{C}_p$ with the action of $\gamma \in \Gamma$ as multiplication by $\varphi(\gamma)$, we have $A_p(\rho_\varphi^{(i)}) =$

$A_p^{(i)} \otimes_\Lambda \mathbf{C}_p(\varphi)$. The next Theorem is the key to bridge between the Alexander module and the cohomology.

Theorem 3.4. *We have an isomorphism*

$$\mathrm{Hom}_{\mathbf{C}_p}(A_p(\rho_\varphi^{(i)}), \mathbf{C}_p) \simeq Z^1(G_p, \mathbf{C}_p(\rho_\varphi^{(i)}))$$

where the r.h.s. denotes the space of 1*-cocycles* $G_p \to \mathbf{C}_p(\rho_\varphi^{(i)})$.

Proof. By Definition 3.1, we have

$$A_p(\rho_\varphi^{(i)}) \simeq \bigoplus_{g \in G_p} \mathbf{C}_p dg / \langle d(fg) - df - \rho_\varphi^{(i)}(f) dg \rangle_{\mathbf{C}_p}.$$

Hence we have

$$\begin{aligned} \mathrm{Hom}_{\mathbf{C}_p}(A_p(\rho_\varphi^{(i)}), \mathbf{C}_p) &= \mathrm{Hom}_{\mathbf{C}_p}(\bigoplus_{g \in G_p} \mathbf{C}_p dg / \langle d(fg) - df - \rho_\varphi^{(i)}(f) dg \rangle_{\mathbf{C}_p}, \mathbf{C}_p) \\ &\simeq Z^1(G_p, \mathbf{C}_p(\rho_\varphi^{(i)})). \quad \square \end{aligned}$$

Corollary 3.5. *We have*

$$\dim H^1(G_p, \mathbf{C}_p(\rho_\varphi^{(i)})) = \begin{cases} \dim A_p(\rho_\varphi^{(i)}) - 1, & \text{if } (i, \varphi) \neq (0, \mathbf{1}) \\ 1, & \text{if } (i, \varphi) = (0, \mathbf{1}) \end{cases}$$

where $\mathbf{1}$ *is the trivial* $\mathbf{Z}_p$*-algebra homomorphism defined by* $\mathbf{1}(\gamma) = 1$.

Proof. We easily see by definition that the space of 1-coboundaries is given by

$$B^1(G_p, \mathbf{C}_p(\rho_\varphi^{(i)})) = \begin{cases} \mathbf{C}_p & \text{if } \rho_\varphi^{(i)} \neq \mathbf{1} \text{ (trivial repr.)} \\ \{0\} & \text{if } \rho_\varphi^{(i)} = \mathbf{1} \end{cases}$$

It is also easy to see that

$$\rho_\varphi^{(i)} = \mathbf{1} \Longleftrightarrow (i, \varphi) = (0, \mathbf{1})$$

and $\dim H^1(G_p, \mathbf{C}_p(\mathbf{1})) = 1$. Hence the assertion follows from Theorem 3.4.
$\square$

4. Proof of the Main Theorem

Let i be an even integer mod $p-1$. By Iwasawa ([Iw]), $H_\infty^{(i)}$ is a finitely generated torsion Λ-module. Let

$$(4.1) \qquad \Lambda^m \overset{P^{(i)}}{\longrightarrow} \Lambda^n \longrightarrow A_p^{(i)} \longrightarrow 0$$

be a resolution of the Λ-module $A_p^{(i)}$, where $P^{(i)}$ denotes the presentation matrix over Λ. By Proposition 3.3, the $(d-1)$-th Iwasawa ideal $J_{p,d-1}^{(i)}$ is the d-th elementary ideal of $A_p^{(i)}$ which is given by the ideal of Λ generated by $(n-d)$-minors of $P^{(i)}$ if $d < n$ and 1 if $d \geq n$. By tensoring (4.1) with $\mathbf{C}_p(\varphi)$ over Λ, we get the exact sequence of $\mathbf{C}_p$-vector spaces

$$\mathbf{C}_p(\varphi)^m \overset{\varphi(P^{(i)})}{\longrightarrow} \mathbf{C}_p(\varphi)^n \longrightarrow A_p(\rho_\varphi^{(i)}) \longrightarrow 0.$$

Hence we have $\dim A_p(\rho_\varphi^{(i)}) = n - \mathrm{rank}(\varphi(M))$. By Cororally 3.5, for $d > 1$, we have

$$\begin{aligned} \dim H^1(G_p, \mathbf{C}_p(\rho_\varphi^{(i)})) \geq d \ &\Leftrightarrow \dim A_p(\rho_\varphi^{(i)}) \geq d+1 \\ &\Leftrightarrow \mathrm{rank}(\varphi(P^{(i)})) \leq n-(d+1) \\ &\Leftrightarrow \text{all } (n-d)\text{-minors of } \varphi(P^{(i)}) = 0 \\ &\Leftrightarrow f(\varphi(\gamma)-1) = 0 \text{ for } f \in J_{p,d-1}^{(i)}, \end{aligned}$$

and for $d = 1$, we have

$$\begin{aligned} &\dim H^1(G_p, \mathbf{C}_p(\rho_\varphi^{(i)})) \geq 1 \\ &\Leftrightarrow \rho_\varphi^{(i)} \neq \mathbf{1} \text{ and } \dim A_p(\rho_\varphi^{(i)}) \geq 2 \text{ or } \rho_\varphi^{(i)} = \mathbf{1} \\ &\Leftrightarrow (i,\varphi) \neq (0,\mathbf{1}) \text{ and } f(\varphi(\gamma)-1) = 0 \text{ for } f \in J_{p,0}^{(i)} \text{ or } (i,\varphi) = (0,\mathbf{1}). \ \square \end{aligned}$$

Remark 4.2. Let G be the Galois group over $\mathbf{Q}$ of the maximal pro-p extension of $\mathbf{Q}(\mu_p)$ unramified outside p. As remarked in 1.3, any deformation $\rho_\varphi^{(i)}$ factors through G and so we may use the smaller G instead of G_p. In fact, we can define the Alexander module $A_\psi(G)$ to be the ψ-differential module over $\tilde{\Lambda}$ attached to $\psi : G \to \Delta \times \Gamma$ and show the same relation between $A_\psi(G)$ and $H^1(G, \mathbf{C}_p(\rho_\varphi^{(i)}))$ as in Theorem 3.4 and Corollary 3.5. Furthermore, since G is a finitely presented profinite group, we can give a presentation matrix for $A_\psi(G)$ in terms of the profinite Fox differential calculus ([Ih], Appendix), just like the case of a knot.

Remark 4.3. M. Kurihara ([K1,2]) gave a beautiful resolution of the Λ-module $H_\infty^{(i)}$ by elaborating the theory of Euler system which relates the higher Iwasawa ideals $J_{p,d}^{(i)}$ with the p-adic L function, a refined form of the Iwasawa main conjecture.

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