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A duality theorem based on triangles separating three convex sets

H. Kawasaki

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Faculty of Mathematics
Kyushu University
Fukuoka, JAPAN

A duality theorem based on triangles separating three convex sets. [1]

Hidefumi Kawasaki[2]

Abstract. Separation theorems play the central role in duality theory. Recently, the author proposed a duality theorem for a three-phase partition problem in [4]. It is based on triangles separating three convex sets. However, the dual problem in [4] includes a variable of the primal problem. The aim of this paper is to remove the variable from the dual problem.

Key words. Duality theorem, Separation theorem, Convex set, Partition problem, Triangle

1. Introduction

The three-phase partition problem is to divide a given domain $\Omega \subset \mathbb{R}^2$ into three subdomains with a triple junction having least interfacial area (Fig.1). Sternberg and Zeimer [6] established the existence of local minimizers to the problem. Ikota and Yanagida [1] investigated not only stability but also instability for stationary curves in terms of the curvature of the boundary $\partial\Omega$.

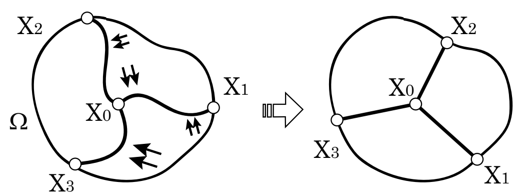

Figure 1. Three-phase partition problem

They formulated the problem as a variational problem. However, since the shortest curve joining two points is the line segment, it can be formulated as an extremal problems in $\mathbb{R}^n$. From this point of view, the author discussed stability and instability of the three-phase partition problem and studied its game-theoretic aspect in [2][3]. Further, he gave a duality theorem for the following problem in [4].

$$(P_0) \qquad \begin{array}{ll} \text{Minimize} & f(X_0,\dots,X_3) := \sum_{i=1}^{3} ||X_i - X_0|| \\ \text{subject to} & X_0 \in \Omega,\ X_i \in C_i\ (i = 1,2,3), \end{array}$$

where $||\cdot||$ denotes the Euclidean norm and C_i $(i = 1,2,3)$ are closed convex sets with non-empty interior in $\mathbb{R}^2$ such that $\Omega := \mathrm{cl}(\cap_{i=1}^{3} C_i^c)$ is non-empty (Fig. 2) .

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[2]Associate Professor, Faculty of Mathematics, Kyushu University, Japan

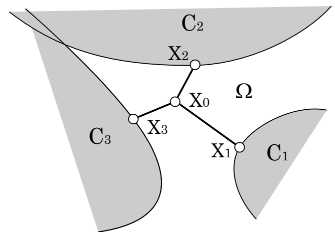

FIGURE 2. Primal problem (P_0)

Let $(X_0, \ldots, X_3)$ be a non-degenerate minimum solution for (P_0), that is, X_0 does not coincide with any X_i $(i = 1, 2, 3)$. Assume that Ω is bounded. Then, for the following dual problem (D_0^*), we have $\min(P_0) = \max(D_0^*)$ ([4]).

$$(D_0^*) \qquad \begin{array}{ll} \text{Maximize} & \text{the smallest height of a triangle } \Delta \\ \text{subject to} & X_0 \in \Delta \subset \Omega. \end{array}$$

The main aim of this paper is to remove X_0 from the dual problem (D_0^*). For this aim, we slightly change the primal problem as follows.

$$(P) \qquad \begin{array}{ll} \text{Minimize} & \displaystyle\sum_{i=1}^{3} ||X_i - X_0|| \\ \text{subject to} & X_0 \in \mathbb{R}^2,\ X_i \in C_i\ (i = 1, 2, 3). \end{array}$$

The only difference between (P) and (P_0) is the domain of X_0. When we emphasize the domain Ω, we denote (P) by (P_Ω).

This paper is organized as follows. In Section 2, we briefly review classical duality theorems and introduce the concept of separation of three convex sets by a triangle. In Section 3, we characterize minimum solutions for (P). In Section 4, we define the dual problem (D) and show duality.

We close this section with our notations. For any closed convex sets C_1 and C_2, we define $d(C_1, C_2) := \min\{||X_1 - X_2|| \mid X_i \in C_i\ (i = 1, 2)\}$. We denote by $N(X_i; C_i)$ the normal cone of C_i at X_i. When $X_i \neq X_0$, we denote by e_i the unit vector $(X_i - X_0)/||X_i - X_0||$.

2. **Separation by a triangle**

In this section, we first review classical duality theorems in brief. Next, we introduce separation of three convex sets by a triangle.

One of the simplest duality theorems is the following. Let C_1 be a non-empty convex set in $\mathbb{R}^2$ and $A \notin C_1$ a point. Then the primal problem is

$$(P_1) \qquad \begin{array}{ll} \text{Minimize} & ||X_1 - A|| \\ \text{subject to} & X_1 \in C_1. \end{array}$$

Its dual problem (D_1) is to maximize the distance from A to a hyperplane H that separates A and C_1. We can rephrase it as maximizing the width of a strip that separates A and C_1 (Fig. 3), where a strip stands for the area sandwiched between two parallel lines.

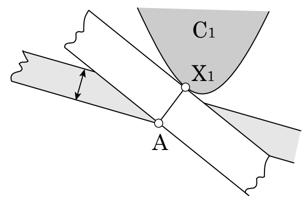

FIGURE 3. Dual problem (D_1)

If we replace A with a convex set C_2 such that $C_1 \cap C_2 = \phi$, then the primal problem is as follows.

$$(P_2) \qquad \begin{array}{ll} \text{Minimize} & ||X_1 - X_2|| \\ \text{subject to} & X_i \in C_i \ (i = 1, 2). \end{array}$$

Its dual problem (D_2) is to minimize the width of a strip that separates C_1 and C_2 (Fig. 4).

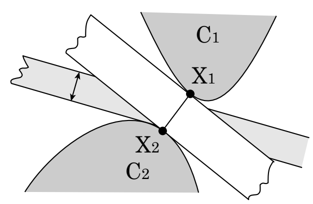

FIGURE 4. Dual problem (D_2)

If we take the epigraph of a convex function f and the hypograph of a concave function g as C_1 and C_2, respectively, and measure the width of the strip in the vertical direction, duality between (P_2) and (D_2) reduces to Fenchel's duality, see e.g. [5, Theorem 31.1].

Therefore, classical dual problems can be described in terms of strips or hyperplanes separating two convex sets. In this paper, we need a concept of triangles separating three convex sets in order to deal with (P).

Definition 2.1. ([4]) Let C_i ($i = 1, 2, 3$) be convex sets in $\mathbb{R}^2$ such that $\Omega = \mathrm{cl}(\cap_{i=1}^3 C_i^c)$ is not empty, and let $\Delta \subset \Omega$ a triangle. Then, we say that Δ separates $\{C_i\}_{i=1}^3$ if there are three closed half spaces $\{H_i^-\}_{i=1}^3$ such that $C_i \subset H_i^-$ for every i and $\Delta = \cap_{i=1}^3 H_i^+$, where H_i^+ denotes the closed half space opposite to H_i^- (Fig. 5).

The following lemma is useful in this paper.

Lemma 2.1. *Let* $(X_0, \ldots, X_3)$ *be a feasible solution for* (P) *and let a triangle* Δ *separate* $\{C_i\}_{i=1}^3$. *Then* $\min(P_\Delta) \leq \sum_{i=1}^3 ||X_i - X_0||$.

Proof. Since $X_i \in C_i \subset H_i^-$, we have $\sum_{i=1}^3 ||X_i - X_0|| \geq \sum_{i=1}^3 d(X_0, H_i^-) \geq \min(P_\Delta)$. □

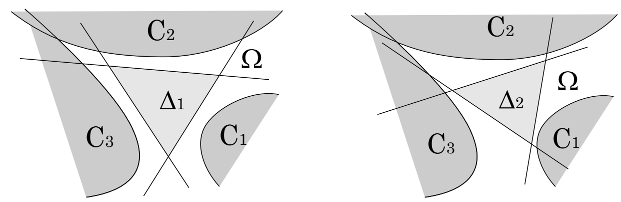

FIGURE 5. Δ_1 separates C_i's and Δ_2 does not separate C_i's.

3. Characterization of minimum solutions

In this section, we first give a characterization theorem of minimum solutions for (P). Next, we consider the special case that C_i's are closed half spaces.

Although (P_0) is not a convex program, the present primal problem (P) is a convex program. So optimal solutions are characterized by the first-order optimality condition below. Since the proof is almost same with [4, Theorem 3.1], we omit the proof.

Theorem 3.1. *Let $(X_0, \ldots, X_3)$ be a non-degenerate feasible solution for (P). Then it is a minimum solution if and only if it satisfies Young's law*

$$\angle X_i X_0 X_j = 120^\circ \text{ for any } i \neq j \tag{3.1}$$

and the transversality condition

$$X_0 - X_i \in N(X_i; C_i) \quad (i = 1, 2, 3). \tag{3.2}$$

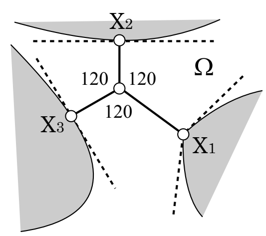

FIGURE 6. Young's law and the transversality condition

Next, we consider the special case that Ω is a triangle determined by closed half spaces C_i $(i = 1, 2, 3)$. Then it is clear that the minimum is attained by $(X_0, \ldots, X_3)$ satisfying $X_0 \in \Omega$. So, (P) reduces to (P_0). Hence, Corollary 1 in [4] is available to (P).

Proposition 3.1. *When Ω is a triangle,* $\min(P)$ *equals to the smallest height of* Ω.

4. Duality theorem

The dual problem (D) is defined as follows.

	Maximize	the smallest height of a triangle Δ
(D)	subject to	there exists a triangle Δ' such that $\Delta \subset \Delta' \subset \Omega$,
		Δ' separates $\{C_i\}_{i=1}^3$.

When Ω is bounded, it has a simplified form (D^*) defined in Theorem 4.2 below.

Theorem 4.1. *Let* (X_0, X_1, X_2, X_3) *and* Δ *be feasible solutions for* (P) *and* (D)*, respectively, then it holds that* $\min(P_\Delta) \le \sum_{i=1}^{3} ||X_i - X_0||$*, so that*

$$\sup(D) \le \inf(P). \tag{4.1}$$

Furthermore, if (P) *has a non-degenerate minimum, then*

$$\min(P_0) = \min(P) = \max(D). \tag{4.2}$$

Proof. Let Δ be a feasible solution for (D). Then there exists a triangle Δ' such that $\Delta \subset \Delta' \subset \Omega$ and Δ' separates $\{C_i\}_{i=1}^3$. Let (X_0, X_1, X_2, X_3) be a feasible solution for (P). Then, combining Lemma 2.1 and Proposition 3.1, we have

$$\min(P_\Delta) \le \min(P_{\Delta'}) \le \sum_{i=1}^{3} ||X_i - X_0||, \tag{4.3}$$

which implies the weak duality (4.1). By Theorem 3.1, the non-degenerate minimum solution forms a regular triangle Δ^* such that

$$\min(P_\Omega) = \text{the height of } \Delta^* = \min(P_{\Delta^*}). \tag{4.4}$$

It follows from definition of the normal cone that Δ^* itself separates $\{C_i\}_{i=1}^3$. Therefore, Δ^* attains the maximum of (D). So we get the strong duality $\min(P) = \max(D)$. On the other hand, since X_0 is in the interior of Ω, there exists a convex neighborhood C_0 of X_0 such that $C_0 \subset \Omega$. Since the primal problem (P_0) restricted on C_0 is a convex program, (X_0, X_1, X_2, X_3) is a minimum solution for (P). $\square$

Theorem 4.2. *When* Ω *is bounded, the dual problem* (D) *is simplified as follows.*

(D^*) *Maximize the smallest height of a triangle* $\Delta \subset \Omega$.

Proof. Assume that $\Delta \subset \Omega$. Then, by separation theorem, there are closed half spaces H_i^- $(i = 1, 2, 3)$ such that $C_i \subset H_i^-$ and $\Delta \subset \cap_{i=1}^3 (H_i^+)^c =: \Delta'$. Since Ω is bounded and since $\Delta' \subset \cap_{i=1}^3 C_i^c = \Omega$, Δ' is a triangle separating $\{C_i\}_{i=1}^3$. $\square$

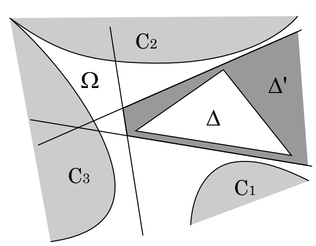

FIGURE 7. Separating hyperplanes form an unbounded polygon Δ'

5. **Concluding remarks**

When Ω is not bounded, separating hyperplanes do not necessarily form a triangle, see Fig. 7. So duality relationship $\min(P) = \max(D^*)$ does not always hold. Indeed, since we can enlarge Δ rightward within the dark gray area as we like, $\sup(D^*)$ equals $+\infty$.

We can replace a triangle by a regular triangle in our dual problems (D) and (D^*), because the maximum is attained by a regular triangle. However, it is clear that regular triangles are not enough when Ω is a (general) triangle. That's why we defined the dual problem with (general) triangles.

References

[1] R. Ikota and E. Yanagida, "A stability criterion for stationary curves to the curvature-driven motion with a triple junction", *Differential and Integral Equations*, 16, 707–726 (2003).

[2] H. Kawasaki, "A game-theoretic aspect of conjugate sets for a nonlinear programming problem", in Proceedings of the third International Conference on Nonlinear Analysis and Convex Analysis, Yokohama Publishers, 159–168 (2004).

[3] H. Kawasaki, "Conjugate-set game for a nonlinear programming problem", in *Game theory and applications 10*, eds. L.A. Petrosjan and V.V. Mazalov, Nova Science Publishers, New York, USA, 87–95 (2005).

[4] H. Kawasaki, "A duality theorem for a three-phase partition problem", submitted.

[5] R.T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, New Jersey, (1970).

[6] P. Sternberg and W. P. Zeimer, "Local minimizers of a three-phase partition problem with triple junctions", *Proc. Royal Soc. Edin.*, 124A, 1059–1073 (1994).

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